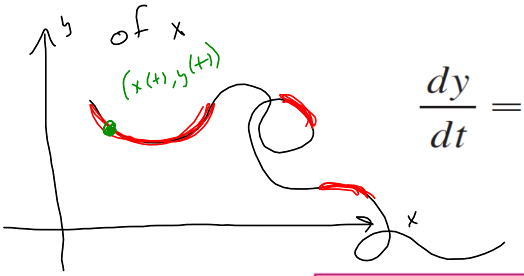
10.2 I - #s 1, 3, 6, 8, 11, 16, 17, 20, 25, 31
10.2 II - #s 34, 37, 38, 43, 51, 57, 58

10.2 Calculus with Parametric Curves

$x = f(t)$, $y = g(t)$. Suppose y is "locally" a function of x



$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Assume $x = x(t)$

$y = 3x$, $x = 5t \Rightarrow y(t) = y(x(t))$

$y'(t) = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Do #s 1, 3, 6, 8

If $\frac{dx}{dt} > 0$

t increasing is moving to the right.

$$\boxed{1} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

We see from that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

It is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$y'' < 0$ $\qquad$ $y'' > 0$

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \quad \boxed{\left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]}$$

$x = f(t) \implies$

$\frac{dx}{dt} = f'(t)$

$dx = f'(t)\,dt$

$\int_{x=a}^{x=b} y\,dx = \int_\alpha^\beta$

$a = f(t)$

$f^{-1}(a) = t = \alpha$

$f^{-1}(b) = \beta$

$\alpha \,\&\, \beta$ switched, when $\frac{dx}{dt} < 0$

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \le x \le b$.

If F' is continuous, then

2 $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

→ ds

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$.

This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

since $x = f(t)$,

$\frac{dx}{dt} = f'(t) \rightarrow dx = f'(t)\,dt = \frac{dx}{dt}\,dt$

This works, even if $y = F(x)$ isn't in the cards. Book goes into detail. I don't.

Since $dx/dt > 0$, we have

3 $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

20

$3 = \sqrt{3^2}$

$\frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2}$, if $\frac{dx}{dt} > 0$

Surface Area

If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

6
$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

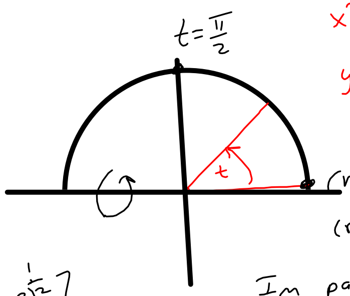
The "parametric" ds

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6

Show that the surface area of a sphere of radius r is $4\pi r^2$.



$x^2+y^2=r^2$

$y=\pm\sqrt{r^2-x^2}$

$=\sqrt{r^2-x^2}$ for top $\frac{1}{2}$

$(r,0) = (r\cos(0), r\sin(0))$

Solution:

The sphere is obtained by rotating the semicircle

$x = r\cos t \qquad y = r\sin t \qquad 0 \le t \le \pi$

about the x-axis.

Im past:

$\frac{d}{dx}\left[(r^2-x^2)^{\frac{1}{2}}\right] = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)$

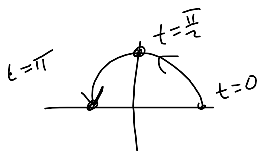
$2\pi\int\sqrt{r^2-x^2}\,ds$

Therefore, from Formula 6, we get

$$S = \int_0^\pi 2\pi r\sin t\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt = 4\pi r^2\text{, obviously!}$$

$ds = \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$

$S = 2\pi\int_a^b y\,ds$



$$S = \int_0^\pi 2\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$$

$$= 2\pi r\int_0^\pi \sin t\sqrt{r^2}\,dt$$

$\sqrt{r^2} = |r| = r$, since $r>0$

$$= 2\pi r\int_0^\pi r\sin t\,dt$$

$$= 2\pi r^2\int_0^\pi \sin t\,dt = -2\pi r^2\Big[\cos t\Big]_0^\pi$$

$$= -2\pi r^2\big[-1-1\big] = 4\pi r^2!$$

1–2 Find dy/dx.

1. $x = t \sin t, \quad y = t^2 + t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \boxed{\frac{2t+1}{\sin t + t\cos t}}$$

$$\frac{dy}{dt} =$$

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = 1 + 4t - t^2, \quad y = 2 - t^3; \quad t = 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3t^2}{4-2t}$$

$x(1) = 1+4-1 = 4 \qquad (x_1, y_1) = (4, 1)$

$y(1) = 2-1^3 = 1$

$$m_{tan} = \left.\frac{dy}{dx}\right|_{t=1} = \frac{-3}{4-2} = \frac{-3}{2} = m$$

Point-Slope

$y = m(x - x_1) + y_1$

$$\boxed{y = -\frac{3}{2}(x-4) + 1}$$

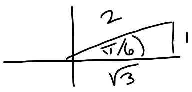
6. $x = \sin^3\theta, \quad y = \cos^3\theta; \quad \theta = \pi/6$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{-3\cos^2\theta\sin\theta}{3\sin^2\theta\cos\theta}\right|_{\theta=\frac{\pi}{6}} = \frac{-3\cos^2(\frac{\pi}{6})\sin(\frac{\pi}{6})}{3\sin^2(\frac{\pi}{6})\cos(\frac{\pi}{6})} = \frac{-3(\frac{\sqrt{3}}{2})^2(\frac{1}{2})}{3(\frac{1}{2})^2(\frac{\sqrt{3}}{2})} = -\frac{\frac{3}{8}}{\frac{3\sqrt{3}}{8}}$$

$$= -\frac{3}{8}\cdot\frac{8}{3\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$x\left(\frac{\pi}{6}\right) = \sin^3\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = x_1$$



$$y\left(\frac{\pi}{6}\right) = \cos^3\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8} = y_1$$

$$y = m(x - x_1) + y_1$$

$$\boxed{y = \frac{1}{\sqrt{3}}\left(x - \frac{1}{8}\right) + \frac{3\sqrt{3}}{8}}$$

d

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

8. $x = 1 + \sqrt{t}$, $y = e^{t^2}$; $(2, e)$

$\sqrt{t} \to t^{\frac{1}{2}}$

(a)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2te^{t^2}}{\frac{1}{2}t^{-1/2}} = 4t^{3/2}e^{t^2}$$

$(x_1, y_1) = (2, e)$

$$\left.\frac{dy}{dx}\right|_{t=1} = 4 \cdot 1 \cdot e = 4e = m$$

Find t:

$x(t) = 1 + \sqrt{t} \overset{SET}{=} 2$

$\sqrt{t} = 1$

$t = 1$

$$\boxed{y = 4e(x-2) + e}$$

(b) $x = 1 + \sqrt{t}$

$1 + \sqrt{t} = x$

$\sqrt{t} = x - 1$

$t = (x-1)^2$

$y = e^{t^2} = e^{((x-1)^2)^2} = e^{(x-1)^4}$

$$\left.\frac{dy}{dx}\right|_{x=2} = \left. 4(x-1)^3 e^{(x-1)^4} \right|_{x=2} = 4e = m$$

$$\boxed{y = 4e(x-2) + e}$$

11–16 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

11. $x = t^2 + 1, \quad y = t^2 + t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t} = 1 + \frac{1}{2t} = \underline{1 + \tfrac{1}{2}t^{-1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} =$$

$\frac{d^2y}{dt\,dx}$ ← Maybe inappropriate in the current context. Save mixed 2nd partials for Calc III

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}t^{-2}}{2t} = -\frac{1}{4}t^{-3}$$

Want > 0 concave upward

$$\Longrightarrow \frac{1}{4}t^{-3} < 0 \Longrightarrow \frac{1}{4t^3} < 0$$

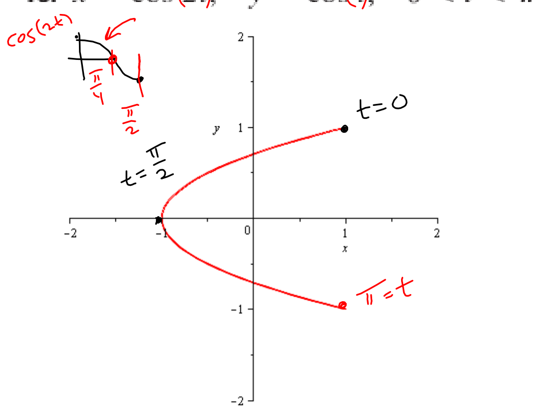
< 0

$-$ $+$

yes 0 No t

$$\boxed{(-\infty, 0)}$$

16. $x = \cos(2t), \quad y = \cos(t), \quad 0 < t < \pi$



$$\frac{dx}{dt} = -2\sin(2t)$$

$$\frac{dy}{dt} = -\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{-2\sin(2t)} = \frac{\sin t}{4\sin t\cos t} = \frac{1}{4}\sec t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{4}\sec t\tan t}{-4\sin t\cos t} = -\frac{1}{16}\,\frac{\sec t\,\frac{\sin t}{\cos t}}{\sin t\cos t}$$

$$= -\frac{1}{16}\,\frac{\sec^2 t\sin t}{\sin t\cos t} = -\frac{1}{16}\sec^3 t$$

want > 0 $\qquad -\frac{1}{16}\sec^3 t$

$-\sec t, -\sec^3 t, -\sec^5 t$

$\cos t$: 0 $+$ $\frac{\pi}{2}$ $-$ $\frac{3\pi}{2}$ $+$; $\frac{\pi}{2} = 0$, $\frac{3\pi}{2} = 0$

$\sec t$: 0 $+$ $\frac{\pi}{2}$ $-$ $\frac{3\pi}{2}$ $+$; $\frac{\pi}{2}$ ∄, $\frac{3\pi}{2}$ ∄

$-\sec t$: 0 $-$ N $\frac{\pi}{2}$ $+$ Y $\frac{3\pi}{2}$ $-$ N; $\frac{\pi}{2}$ ∄ N, $\frac{3\pi}{2}$ ∄ N

$\frac{\pi}{2} < t < \frac{3\pi}{2}$

and $0 < t < \pi$

AND

$0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2}$

$t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cap (0, \pi)$

$= \left(\frac{\pi}{2}, \pi\right)$

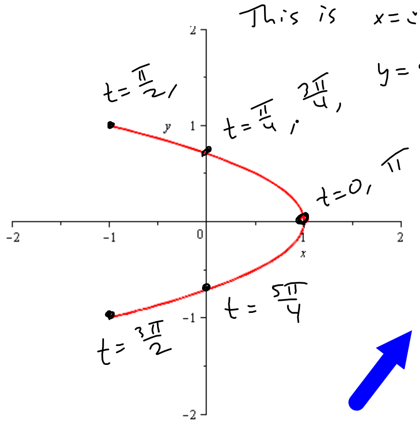
AND !

I'm including this bad attempt as the sort of thing a student ought to keep, with a few comments on how it was messed up.

16. $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$

This is $x = \cos(2t)$

$y = \sin(t)$ Miscopy !



$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{-2\sin(2t)}$$

$$= \frac{1}{2}\,\frac{\sin t}{\sin(2t)} = \frac{dy}{dx}$$

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{\cos(t)\sin(2t) - \sin(t)(2\cos(2t))}{\sin^2(2t)}\right) \quad \text{want} > 0$$

$0 < t < \pi$

$\sin(2t) = 2\sin(t)\cos(t)$

$\cos(2t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2(t)$

$\Rightarrow \cos(t)\sin(2t) - \sin(t)(2\cos(2t)) > 0$

$\Rightarrow 2\sin t\cos^2 t - (\sin t)(2(1-2\sin^2 t)) > 0$

$\Rightarrow 2\sin t\cos^2 t - 2\sin t + 4\sin^3 t > 0$

$\Rightarrow 2\sin t\left[\cos^2 t - 2 + 4\sin^2 t\right]$

$= 2\sin t\left[1 - \sin^2 t - 2 + 4\sin^2 t\right]$ UGLY !

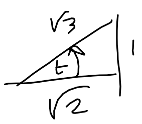
$= 2\sin t\left[-1 + 3\sin^2 t\right]$

$= 2\sin t\left[3\sin^2 t - 1\right] > 0 \Rightarrow$

$\sin t = 0$ $\quad 3\sin^2 t - 1 = 0$

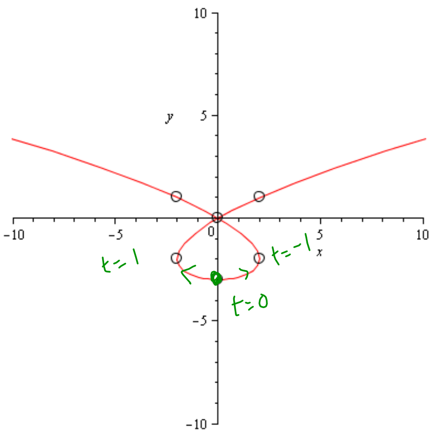
$t = 0, \pi, 2\pi$ $\quad \sin^2 t = \frac{1}{3}$

$\sin t = \pm\frac{1}{\sqrt{3}}$



17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = t^3 - 3t, \quad y = t^2 - 3$



$$\frac{dy}{dx} = \frac{2t}{3t^2-3}$$

① $= 0$

$2t = 0$

$t = 0$

② set $3t^2 - 3 = 0$

$t^2 - 1 = 0$

$t = \pm 1$

$x(0) = 0$
$y(0) = -3$ $\rightsquigarrow$ $(0,-3)$ $\quad m_{tan} = 0$

$x(1) = 1^3 - 3 = -2$
$y(1) = 1^2 - 3 = -2$ $\rightsquigarrow$ $(-2,-2)$

$(x(-1), y(-1)) = (2,-2)$

$m_{tan} \nexists$

20. $x = e^{\sin\theta}, \quad y = e^{\cos\theta}$

$\dfrac{dy}{dt} = -\sin\theta\, e^{\cos\theta}$ HOR SET $= 0 \Rightarrow -\sin\theta\, e^{\cos\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pm\pi, \pm 2\pi, \ldots$

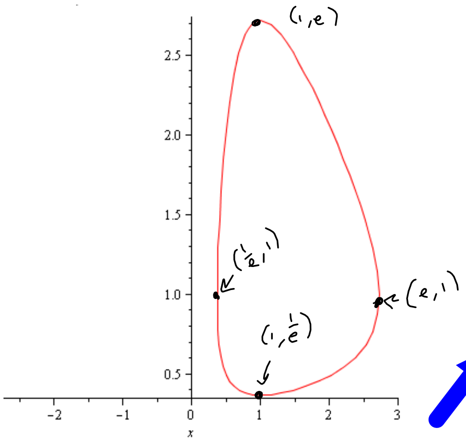
$\dfrac{dx}{dt} = \cos\theta\, e^{\sin\theta}$ VERT SET $0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

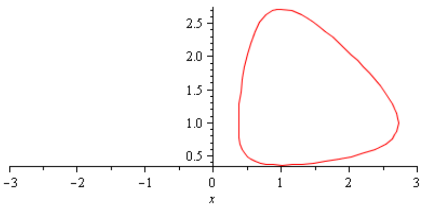
$\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, \ldots$

$(2(-1)+1)\frac{\pi}{2} = -\frac{\pi}{2}$

$\theta = 0 \quad (e^0, e^1) = (1, e)$

$\theta = \frac{\pi}{2} \quad (e^1, e^0) = (e, 1)$





#20

$myx := t \rightarrow e^{\sin(t)}$

$t \rightarrow e^{\sin(t)}$

$myy := t \rightarrow e^{\cos(t)}$

$t \rightarrow e^{\cos(t)}$

$[[myx(0), myy(0)], [myx(\mathrm{Pi}), myy(\mathrm{Pi})], [myx(2\cdot\mathrm{Pi}), myy(2\cdot\mathrm{Pi})], [myx(-\mathrm{Pi}), myy(-\mathrm{Pi})], [myx(-2\cdot\mathrm{Pi}), myy(-2\cdot\mathrm{Pi})], \left[myx\left(\frac{\mathrm{Pi}}{2}\right), myy\left(\frac{\mathrm{Pi}}{2}\right)\right], \left[myx\left(-\frac{\mathrm{Pi}}{2}\right), myy\left(-\frac{\mathrm{Pi}}{2}\right)\right], \left[myx\left(\frac{3\cdot\mathrm{Pi}}{2}\right), myy\left(\frac{3\cdot\mathrm{Pi}}{2}\right)\right]]$

$[[1, e], [1, e^{-1}], [1, e], [1, e^{-1}], [1, e], [e, 1], [e^{-1}, 1], [e^{-1}, 1]]$

$plot([myx(t), myy(t), t = 0..2\cdot\mathrm{Pi}], x = -3..3)$

25. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

So $(x,y) = (0,0)$ at more than one corresponding value of t (Better!

It's periodic!)

$n = 0, \pm 1, \pm 2, \ldots$ = "$n \in \mathbb{Z}$"

$x = \cos(t) \overset{SET}{=} 0 \Rightarrow t = (2n+1)\cdot\frac{\pi}{2}$

$y = \sin t \cos t \overset{SET}{=} 0 \Rightarrow$

$\sin(t) = 0$ OR $\cos(t) = 0$ DONE

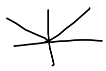
$t = n\pi,\ n \in \mathbb{Z}.$

$2\sin^2 t = 1$

$\sin = \pm\frac{1}{\sqrt{2}}$

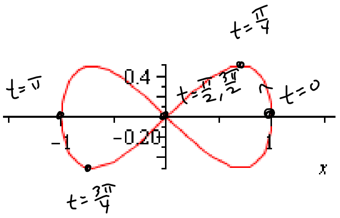
$\frac{dy}{dt} = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t$

$t = \frac{\pi}{2},\ t = \frac{3\pi}{2}$



$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{1 - 2\sin^2(t)}{-\sin t} = y'$



$y'\left(\frac{\pi}{2}\right) = \frac{1 - 2(1)^2}{-1} = +1 \Rightarrow \boxed{y = x}$

$y'\left(\frac{3\pi}{2}\right) = \frac{1 - 2(-1)^2}{-(-1)} = -1 \Rightarrow \boxed{y = -x}$

$\cos t = -1$

$t = \frac{3\pi}{2}, -\frac{\pi}{2}$

31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

End of §10.2 I

$y = f(x)$

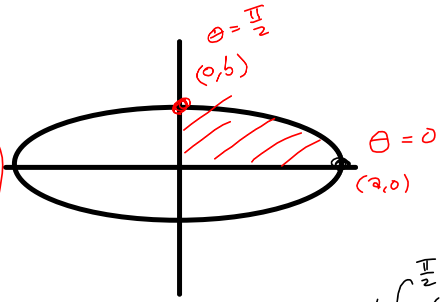
$$\text{Area} = \int_{x_1}^{x_2} y\,dx$$

$$= \int_\alpha^\beta g(t)f'(t)\,dt \quad \left(\text{or } \int_\beta^\alpha\right)$$

where

$y = g(t) = b\sin\theta$

$x = f(t) = a\cos\theta \Rightarrow f'(t)\,dt = -a\sin\theta\,d\theta$



$$4\int_0^{\frac{\pi}{2}} b\sin\theta(-a\sin\theta\,d\theta)$$

$$= -4ab\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$

$$= -4ab\int_0^{\frac{\pi}{2}} \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= -2ab\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \cos(2\theta)\right)d\theta$$

$$= -2ab\left[\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$= -2ab\left[\frac{\pi}{2} - 0\right] = -\pi ab$$

$\frac{dx}{d\theta} < 0 \Rightarrow \int_{\frac{\pi}{2}}^{0}$

Actual $= \pi ab$

We change/switch the limits of integration to switch the sign.

$$\int_0^a y\,dx = \int_{\frac{\pi}{2}}^0$$

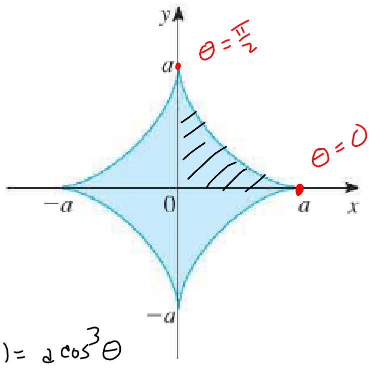
$x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$x = a \Rightarrow \theta = 0$

$\Big\}$ Substitution, going from x to θ as the variable.

Commence $\frac{\pi}{}$

34. Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Laboratory Project on page 668.)



$x = f(\theta) = a\cos^3\theta$

$dx = 3a\cos^2\theta \cdot (-\sin(\theta))\,d\theta$

$*$ $\frac{1-\cos(2\theta)}{2} \cdot \frac{1+\cos(2\theta)}{2}$

$= \frac{1}{4}(1-\cos^2(2\theta))$

$= \frac{1}{4}(\sin^2(2\theta))$

Area $= 4 \cdot$ shaded

$= 4\int_0^a y\,dx$

$= 4\int_{\frac{\pi}{2}}^{0} a\sin^3\theta\left(-3a\cos^2\theta\sin\theta\,d\theta\right)$

$= -12a^2\int_{\frac{\pi}{2}}^{0} \sin^4\theta\cos^2\theta\,d\theta$

$= -12a^2\int_{\frac{\pi}{2}}^{0} \sin^2\theta\left(\frac{1}{4}\sin^2(2\theta)\right)d\theta$ $*$!?

$= -12a^2\int_{\frac{\pi}{2}}^{0} \frac{1-\cos(2\theta)}{2}\cdot\frac{1}{4}\sin^2(2\theta)\,d\theta$

$= -\frac{12}{8}a^2\int_{\frac{\pi}{2}}^{0} (1-\cos(2\theta))(\sin^2(2\theta))\,d\theta$

$= -\frac{3}{2}a^2\left[\int_{\frac{\pi}{2}}^{0}\sin^2(2\theta)\,d\theta - \int_{\frac{\pi}{2}}^{0}\sin^2(2\theta)\cos(2\theta)\,d\theta\right]$

$= -\frac{3}{2}a^2\int_{\frac{\pi}{2}}^{0}\frac{1}{2}(1-\cos(4\theta))\,d\theta + \frac{3}{2}a^2\left[\frac{1}{3}\sin^3(2\theta)\right]_{\frac{\pi}{2}}^{0} \to 0$

$= -\frac{3}{4}a^2\left[\theta - \frac{1}{4}\sin(4\theta)\right]_{\frac{\pi}{2}}^{0}$ $\to 0$

$= -\frac{3}{4}a^2\left[0 - \left(\frac{\pi}{2}\right)\right]$

$= \boxed{\frac{3}{8}\pi a^2}$

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t + e^{-t}, \quad y = t - e^{-t}, \quad 0 \leqslant t \leqslant 2$

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b ds$$

$$\frac{dx}{dt} = 1 - e^{-t} \qquad \frac{dy}{dt} = 1 + e^{-t}$$

$$\left(\frac{dx}{dt}\right)^2 = 1 - 2e^{-t} + e^{-2t} \qquad (e^{-t})^2 = e^{-2t}$$

$$+ \quad \left(\frac{dy}{dt}\right)^2 = 1 + 2e^{-t} + e^{-2t}$$

$$2 + 2e^{-2t}$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{2 + 2e^{-2t}}\, dt$$

$$= \sqrt{2} \int_0^2 \sqrt{1 + (e^{-t})^2}\, dt$$

$$\approx 3.141560426$$

38. $x = t^2 - t, \quad y = t^4, \quad 1 \leqslant t \leqslant 4$

$$\frac{dx}{dt} = 2t - 1 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 4t^2 - 4t + 1$$

$$\frac{dy}{dt} = 4t^3 \Rightarrow \left(\frac{dy}{dt}\right)^2 = 16t^6 \Rightarrow$$

$$S = \int_1^4 \sqrt{4t^2 - 4t + 1 + 16t^6}\, dt \approx 255.3756401 \approx 255.3756$$

41–44 Find the exact length of the curve.

43. $x = t\sin t, \quad y = t\cos t, \quad 0 \leqslant t \leqslant 1$

$$\frac{dx}{dt} = \sin t + t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$$

$$+$$

$$\frac{dy}{dt} = \cos t - t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t$$

$$1 + t^2$$

$$S = \int_0^1 \sqrt{t^2+1}\,dt = \#21 \text{ Formula} = \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}-1)$$

Book Answer: $\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)$

$$\ln(\sqrt{2}-1) = \ln\left((\sqrt{2}-1)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right) = \ln\left(\frac{2-1}{\sqrt{2}+1}\right) = \ln\left(\frac{1}{\sqrt{2}+1}\right) = \ln\left((\sqrt{2}+1)^{-1}\right) = -\ln(\sqrt{2}+1)$$

51–52 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

51. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

$\sin(2x) = 2\sin x \cos x$

$$\frac{dy}{dt} = 2\cos(t)\cdot(-\sin(t)) = -2\sin(t)\cos(t) \Rightarrow \left(\frac{dy}{dt}\right)^2 = 4\sin^2(t)\cos^2(t) = (2\sin t\cos t)^2 = (\sin(2t))^2 = \sin^2(2t)$$

$$\frac{dx}{dt} = 2\sin(t)\cos(t) \Rightarrow \left(\frac{dx}{dt}\right)^2 = 4\sin^2(t)\cos^2(t)$$

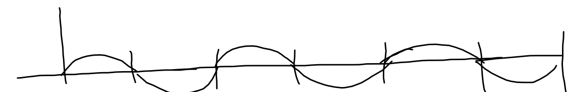
$$\Rightarrow ds = \sqrt{4\sin^2 t\cos^2 t + 4\sin^2 t\cos^2 t} = \sqrt{8\sin^2 t\cos^2 t}$$

$$= \sqrt{2\,(2^2\sin^2 t\cos^2 t)} = \sqrt{2(2\sin t\cos t)^2}$$

$$= \sqrt{2(\sin(2t))^2} = \sqrt{2\sin^2(2t)} = \sqrt{2}\,|\sin(2t)|$$

$$\Rightarrow \text{Distance} = \int_0^{3\pi} \sqrt{2}\,|\sin(2t)|\,dt$$

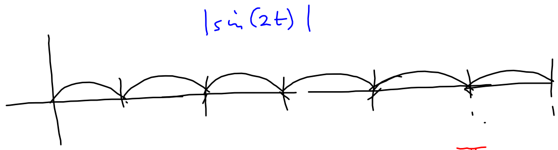
$\sin t$ $\qquad$ $\sin(2t)$



For $\sin(t)$: π $\quad 2\pi$ $\quad 3\pi$ $\quad 4\pi$ $\quad 5\pi$ $\quad 6\pi$

For $\sin(2t)$: $\frac{\pi}{2}$ $\quad \pi$ $\quad \frac{3\pi}{2}$ $\quad 2\pi$ $\quad \frac{5\pi}{2}$ $\quad 3\pi$

$|\sin(2t)|$



$\frac{\pi}{2}$ $\quad \pi$ $\quad \frac{3\pi}{2}$ $\quad 2\pi$ $\quad \frac{5\pi}{2}$ $\quad 3\pi$

$$\therefore \sqrt{2}\int_0^{3\pi} |\sin(2t)|\,dt = 6\sqrt{2}\int_0^{\frac{\pi}{2}} |\sin(2t)|\,dt = 6\sqrt{2}\int_0^{\frac{\pi}{2}} \sin(2t)\,dt$$

$$= (6\sqrt{2})\left(\tfrac{1}{2}\right)\int_0^{\frac{\pi}{2}} \sin(2t)\,(2dt) = 3\sqrt{2}\left[-\cos(2t)\right]_0^{\frac{\pi}{2}}$$

$$= -3\sqrt{2}\left[\cos(\pi) - \cos(0)\right] = \boxed{6\sqrt{2} = \text{Distance}}$$

Arc Length is a different matter

$x + y = \sin^2(t) + \cos^2(t) = 1$

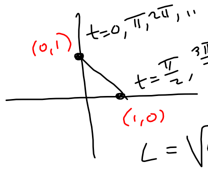
$x + y = 1$ and this thing repeats its route

$t = 0 \Rightarrow$

$x = 0, y = 1$

$t = \frac{\pi}{2} \Rightarrow$

$x = 1, y = 0$



$$L = \sqrt{(1-0)^2 + (0-1)^2} = \boxed{\sqrt{2} = L}$$

or just $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{2}\,|\sin(2t)|\,dt$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} (\sin(2t))(2dt) = \frac{\sqrt{2}}{2}\left[-\cos(2t)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}}{2}\left[-\cos(\pi) - (-\cos(0))\right]$$

$$= \frac{\sqrt{2}}{2}\left[-(-1) + 1\right] = \frac{\sqrt{2}}{2}(2) = \sqrt{2}$$

57–60 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

Forgot the 2π!
0.7542998137 $\approx$ .7543

57. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

$$2\pi \int_a^b y \, ds = 2\pi \int_0^{\frac{\pi}{2}} t \cos t \sqrt{t^2+1} \, dt \approx 4.739405508 \approx \boxed{4.7394}$$

$$\frac{dx}{dt} = \sin t + t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$$

$t^2(\cos^2 t + \sin^2 t) = t^2$

$$\frac{dy}{dt} = \cos t + (t(-\sin t))$$

$$= \cos t - t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t$$

$$1 \quad + \quad t^2$$

58. $x = \sin t, \quad y = \sin 2t, \quad 0 \leqslant t \leqslant \pi/2$

$$\frac{dx}{dt} = \cos t \implies \left(\frac{dx}{dt}\right)^2 = \cos^2 t$$

$$\frac{dy}{dt} = 2\cos(2t) \implies \left(\frac{dy}{dt}\right)^2 = 4\cos^2(2t)$$

$$\implies ds = \sqrt{\cos^2(t) + 4\cos^2(2t)}\, dt$$

$$\implies S = 2\pi \int_a^b y\, ds = 2\pi \int_0^{\frac{\pi}{2}} \sin(2t)\sqrt{\cos^2(t) + 4\cos^2(2t)}\, dt$$

$$= 2\pi\left(\frac{17}{64}\sqrt{5} - \frac{31}{1024}\ln(31) + \frac{31}{512}\ln\left(17\sqrt{31} + 8\sqrt{31}\sqrt{5}\right) + \frac{15}{32}\right)$$

$$\approx \boxed{8.028513782 \approx S} \approx \boxed{8.0285}$$