Exponential (unihibited) growth model:

$$\frac{dP}{dt} = kP$$

Rate of change is proportional to size:

$$\boxed{1} \quad \frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k\,dt$$

Assume $P \geq 0$

$$\ln|P| = kt + \hat{C}$$

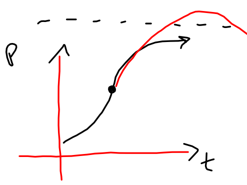
Verhulst, Lotke, Volterra with the "limited growth" model, called the Logistic model, based on this very simple idea.

$$\ln P = kt + \hat{C}$$

$$P = e^{kt+\hat{C}} = e^{kt}e^{\hat{C}}$$

$$= e^{kt} \cdot C$$

$$= \boxed{Ce^{kt}} \ \forall\ C \in \mathbb{R}$$



$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

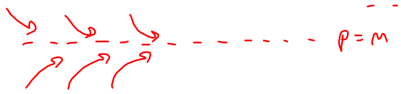
M = carrying capacity.

Equilibrium Solution - A constant solution, basically. Some *y*-value such that any solution hitting that y-value will *stay* at that y-value

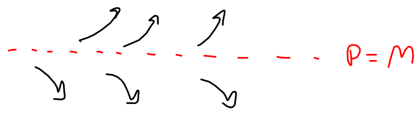
Two Types:

P=M, P=0, for Logistic

Stable



Unstable



Spring Model, Hooke's Law

$x = 0$ @ rest

restoring force $= -kx$

$x =$ how far it's stretched

3rd order: $\frac{d^3x}{dt^2}$

$$\boxed{3} \qquad m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

2nd - order

Ordinary Differential eq'n.

$\frac{d^2x}{dt^2}$

$y'' = -\frac{k}{m}x$

$y' = \int$

$y = \iint$

1. Show that $y = \frac{2}{3}e^{x} + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^{x}$.

$$y' = \frac{2}{3}e^{x} - 2e^{-2x}$$

$$y' + 2y = \frac{2}{3}e^{x} - 2e^{-2x} + 2\left(\frac{2}{3}e^{x} + e^{-2x}\right)$$

$$= \frac{2}{3}e^{x} - 2e^{-2x} + \frac{4}{3}e^{x} + 2e^{-2x}$$

$$= \frac{6}{3}e^{x} = 2e^{x} \checkmark$$

$(fg)' = f'g + fg'$

2. Verify that $y = -t\cos t - t$ is a solution of the initial-value problem

$$ty' = t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$$

$y' = -\cos t + t\sin t - 1$

$ty' = -t\cos t + t^2 \sin t - t \overset{?}{=} y + t^2 \sin t \ ?$

$= \underbrace{-t\cos t - t}_{= \ y} + t^2 \sin t$

$= y + t^2 \sin t$ Yes!

9. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing? (a) $0 < P < 4200$ RHS > 0

(b) For what values of P is the population decreasing? (b) $P > 4200$ RHS < 0

(c) What are the equilibrium solutions?

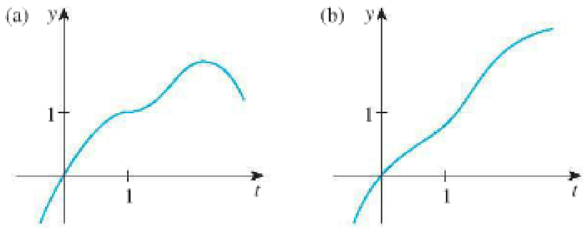
(c) $P = 0$, $P = 4200$ RHS $= 0$

4. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?

(b) For those values of k, verify that every member of the family of functions $y = A \sin kt + B \cos kt$ is also a solution.

11. Explain why the functions with the given graphs *can't* be solutions of the differential equation

$$\frac{dy}{dt} = e^t(y - 1)^2$$



(a) $e^t(y-1)^2 \geq 0$

But graph does

(b) $y=1 \Rightarrow y'=0$, but

$y'>0$ when $y=1$ in graph

15. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.

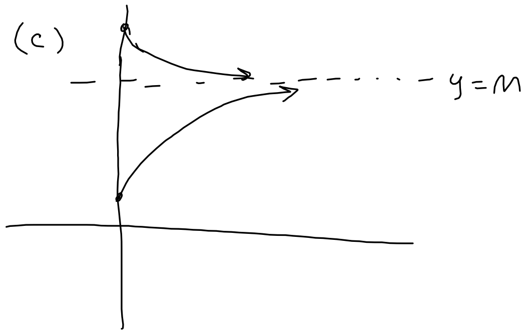
(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

(c) Make a rough sketch of a possible solution of this differential equation.

(c)



(a) when P is small, is when it's increasing most rapidly
$P' = k(M-P) > 0$ when P is small and shrinks as P grows

(b) Model is reasonable, because it slows down when P gets bigger.
More advanced stuff is tougher.