§ 8.5 Cont'd
Random Variables

Probability Distribution

$\boxed{1}$ $$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

$$\sum_{k=1}^{n} p_i = 1$$

$\boxed{2}$ $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$E(x) = \sum x_k p(x_k)$

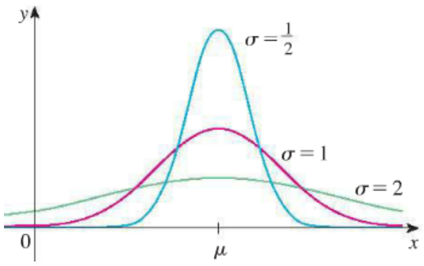
probability density function.

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\,f(x)\,dx}{\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1,\ \text{if } f \text{ is probability}}} = \int_{-\infty}^{\infty} x\,f(x)\,dx = \mu$$

p_k	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36} = \frac{1}{6}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
	2	3	4	5	6	7	8	9	10	11	12

(1,6); (2,5); (3,4)

$\boxed{3}$ **normal distribution**: $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$



Standard Normal: $\sigma = 1$
$\mu = 0$

$$Z(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

1. Let $f(x)$ be the probability density function for the lifetime of a manufacturer's highest quality car tire, where x is measured in miles. Explain the meaning of each integral.

(a) $\int_{30{,}000}^{40{,}000} f(x)\,dx$ (b) $\int_{25{,}000}^{\infty} f(x)\,dx$

(a) This is the probability that a tire lasts between 30,000 and 40,000 miles

(b) This is the probability that a tire lasts more than 25,000 miles.

2. Let $f(t)$ be the probability density function for the time it takes you to drive to school in the morning, where t is measured in minutes. Express the following probabilities as integrals.

(a) The probability that you drive to school in less than 15 minutes

(b) The probability that it takes you more than half an hour to get to school

(a) $\int_{-\infty}^{15} f(t)\,dt$

(b) $\int_{30}^{\infty} f(t)\,dt$

I'd guess

$f(x) \equiv 0$ if $x \leq 0$.

3. Let $f(x) = 30x^2(1 - x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.
(a) Verify that f is a probability density function.
(b) Find $P(X \leqslant \frac{1}{3})$.

(a) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 30x^2(1-x)^2\,dx = 30\int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$f(x) \geq 0$ $\forall x$.

$$= 30\left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\Big|_0^1 = 30\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 30\left(\frac{10 - 15 + 6}{30}\right) = 1 \checkmark$$

(b) $$P(x \leq \tfrac{1}{3}) = \int_{-\infty}^{\frac{1}{3}} f(x)\,dx = \int_0^{\frac{1}{3}} 30x^2(1-x)^2\,dx = \frac{17}{81} \approx 0.2098765432$$

4. Let $f(x) = xe^{-x}$ if $x \geqslant 0$ and $f(x) = 0$ if $x < 0$.
(a) Verify that f is a probability density function.
(b) Find $P(1 \leqslant X \leqslant 2)$.

Not assigned

$$\int_{-\infty}^{\infty} xe^{-x}\,dx = \int_0^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-te^{-t} - e^{-t}\right) - \left[0e^{-0} - e^{-0}\right]$$

$$= 0 - (-1) = 1$$

$$\int xe^{-x}\,dx = uv - \int v\,du = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$u = x$ $\quad dv = e^{-x}\,dx$

$du = dx$ $\quad v = -e^{x}$

$\boxed{f(x) \geq 0 \ \forall x}$

5. Let $f(x) = c/(1 + x^2)$.

(a) For what value of c is f a probability density function?

(b) For that value of c, find $P(-1 < X < 1)$.

(a) $\int_{-\infty}^{\infty} f(x)\,dx \overset{SET}{=} 1$ & solve for c.

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = \lim_{t\to\infty} \int_{-t}^{t} \sim = \lim_{t\to\infty} 2\int_{0}^{\infty} \frac{c}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} 2c \arctan(x)\Big]_0^t = 2\lim_{t\to\infty} c\arctan(t) = 2c\,\frac{\pi}{2} = c\pi \overset{SET}{=} 1 \Rightarrow \boxed{c = \frac{1}{\pi}}$$

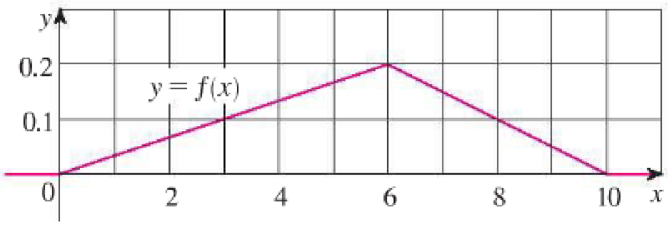
(b) $$\frac{1}{\pi}\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{2}{\pi}\arctan(x)\Big]_0^1 = \boxed{\frac{2}{\pi}\arctan(1)}$$

8. (a) Explain why the function whose graph is shown is a probability density function.

(b) Use the graph to find the following probabilities:

(i) $P(X < 3)$ (ii) $P(3 \leqslant X \leqslant 8)$

(c) Calculate the mean.



15. The speeds of vehicles on a highway with speed limit 100 km/h are normally distributed with mean 112 km/h and standard deviation 8 km/h.

(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed?

(b) If police are instructed to ticket motorists driving 125 km/h or more, what percentage of motorists are targeted?