Hydrostatic Pressure and Force

§8.3 #5, 8, 25, 29, 31

$F = ma = mg = \rho g A d$

from $m = \rho A d$ = Mass of water above the area A

ρd = mass per unit area, like kg/m^2 or lb/in^2

$g = a$ = acceleration due to gravity.

Pressure is Force per unit area $= \frac{F}{A} = \rho g d$

S.I. units : $\frac{N}{m^2} = 1\ Pa = 1$ Pascal

m = meters, not mass

kPa often used:

$1\ kPa = 1{,}000\ Pa.$

$\rho = \frac{kg}{m^3}$

$d = m$

$\rho \cdot d = \frac{kg}{m^3} \cdot m = \frac{kg}{m^2}$

$\rho g d = \frac{kg}{m^2} \cdot \frac{m}{s^2}$ & $F/A = \frac{\rho g A d}{A} =$

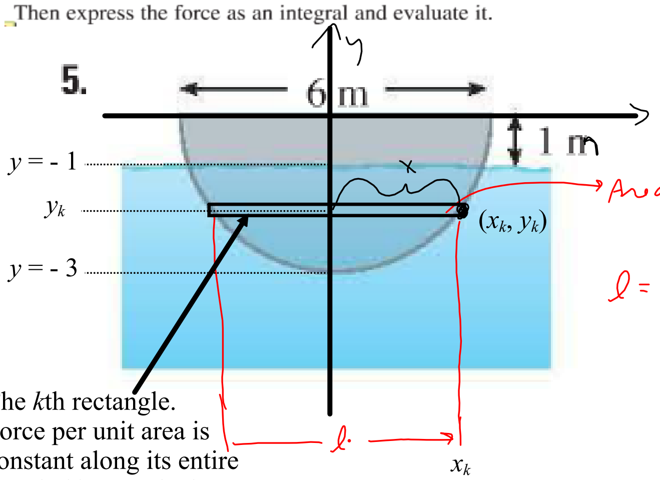
3–11 A vertical plate is submerged (or partially submerged) in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

$\rho g A d$

$x^2 + (y-1)^2 = 3^2$

$\rho = \frac{1000\ kg}{m^3}$

$g = -\frac{9.8\ m}{s^2}$



Area $= 2\sqrt{9-y^2}\ dy$

$\ell = 2x = 2\sqrt{9-y^2}$

Express the area A of the kth rectangle. Easiest way is to plonk the center of the circle at the origin.

$A_k = 2\sqrt{9 - y_k^2}\ \Delta y$

$x^2 + y^2 = r^2$

$x = \pm\sqrt{r^2 - y^2}$

want right half

$x = \sqrt{3^2 - y^2}$

$\ell = 2x = 2\sqrt{9 - y_k^2}$

The kth rectangle. Force per unit area is constant along its entire length, if we make it skinny enough, and the integral's all about skinny rectangles.

$\rho g A d$ is the force on an area A, at a fixed depth, d. Now allow for changes in depth, and changes in cross-section at n different depths. Try to describe the kth rectangle.

$=$ It's nice to treat the depth as the independent variable, because the force per unit area is constant at that depth.

But then, we describe the width at a given depth in terms of the circle, and the easiest way to express the circle is if you put its center at the origin. This makes a discrepancy between depth and y-value, which is easily handled.

Begin: $\sum_{k=1}^{n} \rho\, g\, d_k A_k$

$d_k = -(y_k + 1)$ $\quad d_k = -y_k - 1$

$2 = -(-3) - 1$

$= 3 - 1$

$$= \sum_{k=1}^{n} (9800)\cdot(-1)\cdot(y[k]+1)\cdot 2\cdot\left(\sqrt{9 - y[k]^2}\right)\cdot \Delta y$$

$$= \sum_{k=1}^{n} \left(-19600\,(y_k+1)\sqrt{9-y_k^2}\ \Delta y\right)$$

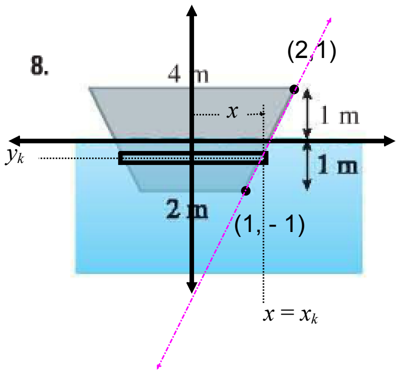
$\rho\, g\, d_k A_k$

$n \to \infty$, $\Delta y \to 0$

$$\int_{-3}^{-1} (-19600)\cdot(y+1)\cdot\left(\sqrt{9-y^2}\right)\, dy$$

$$= -78400\sqrt{3}\sqrt{\pi}\left(-\frac{25}{36}\frac{\sqrt{6}}{\sqrt{\pi}} + \frac{9}{4}\frac{\sqrt{3}\arcsin\left(\frac{1}{3}\sqrt{3}\right)}{\sqrt{\pi}}\right) + 78400\sqrt{3}\sqrt{\pi}\left(-\frac{1}{6}\frac{\sqrt{6}}{\sqrt{\pi}} + \frac{3}{2}\frac{\sqrt{3}\arcsin\left(\frac{1}{3}\sqrt{3}\right)}{\sqrt{\pi}}\right)$$

$\approx$ evalf(%) $= 66980.42295$



I chose the water line as the x-axis, and the horizontal midpoint of the trapezoid as the y-axis.

As before, the object of the picture is to express the area of a representative rectangle of uniform depth on which the pressure is constant per unit area, because the rectangle is *really skinny*. We find the pressure on the plate by adding up the forces on all such rectangles.

With coordinate system in place I built the equation of the line describing the right edge of the trapezoid.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{2-1} = \frac{2}{1} = 2$$
$$y = m(x - x_1) + y_1 = 2(x-1) + (-1)$$
$$= 2x - 2 - 1 = 2x - 3$$
$$y = 2x - 3$$

$$y = 2x - 3$$

We're going to need x as a function of y to integrate with respect to depth, essentially:

$$\Rightarrow 2x - 3 = y$$
$$\Rightarrow 2x = y + 3$$
$$\Rightarrow x = \frac{y+3}{2}$$

The area A of the kth rectangle is $l\,dy$, where l = the length of that kth rectangle and $l = 2x$, from the picture.

We use the work on the left to re-write the area A of that rectangle as

$$A = 2x\ dy = 2\left(\frac{y+3}{2}\right)dy = (y+3)dy$$

Building this thing:

$\rho g A d$

We've built what we need, except for d, in our expression for force on the representative rectangle. It's nice to keep forces positive, and treat depth as a positive number. If I do that, then we see that

From $d = 2$, when $y = -3$,
and $d = 0$, when $y = -1$,
$d = -y - 1$, or $-(y+1)$

$$d_k = -y_k$$

That's the last thing we need to express the force on the kth rectangle.

$\Rightarrow$ The force on the plate is approximated by

$$F \approx -9800 \sum_{k=1}^{n} (y_k + 3)(y_k \cancel{+1})\Delta y$$

$$\xrightarrow{n\to\infty} -9800 \int_{-3}^{-1} (y+3)(y \cancel{+1})dy = F = \text{the EXACT force.}$$

$$\frac{K \cdot m}{s^2}$$

$$F = -9800 \int_{-1}^{0} (y+3)(y)\,dy$$

$$-9800 \cdot \int_{-1}^{0} (y+3)\cdot(y)\,dy$$

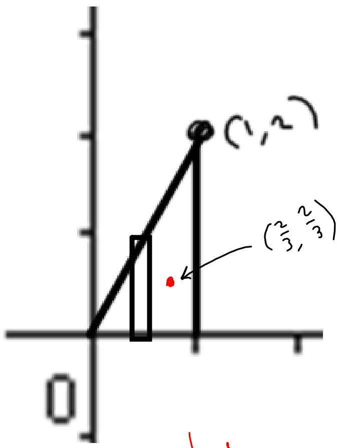
$$= \frac{34300}{3}$$

evalf(%)

$$\approx 11433.33333$$

25–28 Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

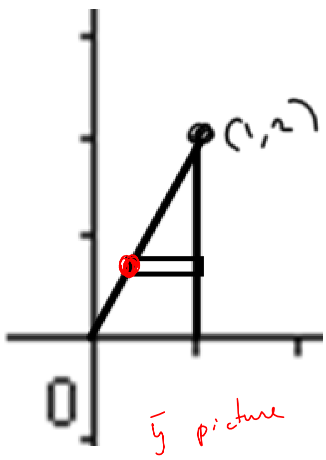
25. $y = 2x, \quad y = 0, \quad x = 1$



$\bar{x}$ picture

$$\bar{x} = \frac{\int_a^b x\,f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_0^1 x \cdot 2x\,dx}{\int_0^1 2x\,dx} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

$$y = 2x \Rightarrow x = \frac{1}{2}y$$



$\bar{y}$ picture

$$\bar{y} = \frac{\int_a^b y\,g(y)\,dy}{\int_a^b g(y)\,dy} = \frac{\int_0^2 y \cdot \left(\frac{1}{2}y\right)dy}{\int_0^2 y \cdot \left(\frac{1}{2}y\right)dy} = \frac{\frac{4}{3}}{1} = \frac{4}{3}$$

I don't like it.

Better

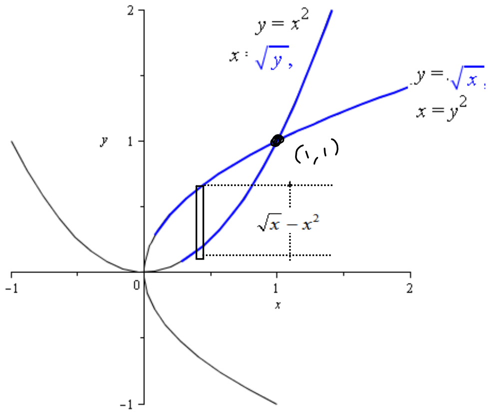
$g(y) = 1 - \frac{1}{2}y$

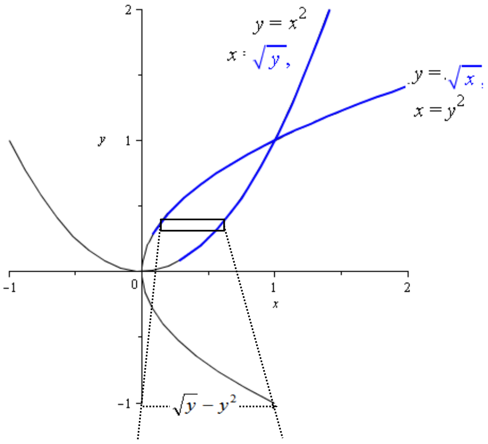
$$\bar{y} = \frac{\int_0^2 y \cdot \left(1 - \frac{1}{2} \cdot y\right) dy}{\int_0^2 \left(1 - \frac{1}{2} \cdot y\right) dy} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

$(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{2}{3}\right)$

29–33 Find the centroid of the region bounded by the given curves.

29. $y = x^2, \quad x = y^2$





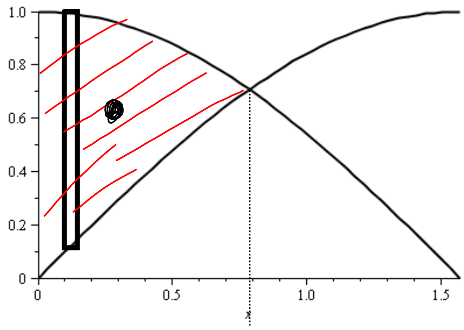
$$\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_0^1 x(\sqrt{x} - x^2)\,dx}{\int_0^1 (\sqrt{x} - x^2)\,dx} = \frac{9}{20}$$

$\hookrightarrow \frac{1}{3}$

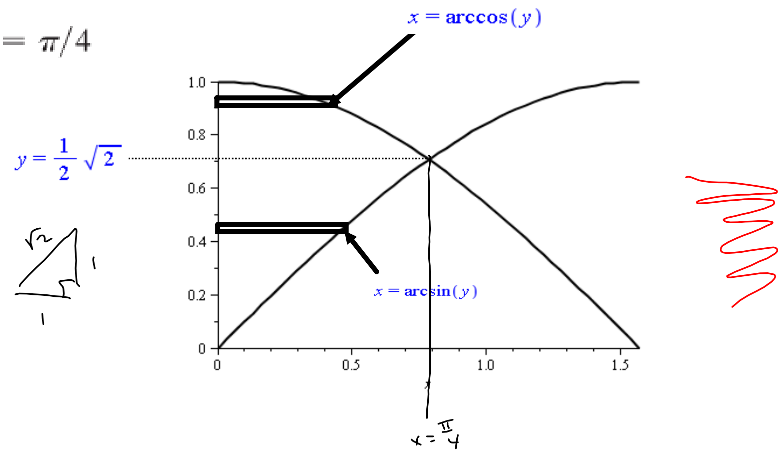
$$(\bar{x}, \bar{y}) = \left(\frac{9}{20}, \frac{9}{20}\right)$$

$$\bar{y} = \frac{\int_0^1 y(\sqrt{y} - y^2)\,dy}{\int_0^1 (\sqrt{y} - y^2)\,dy} = \frac{9}{20}$$

31. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \pi/4$



$x = \pi/4$



$$\bar{x} = \frac{\int_a^b x\, f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_0^{\frac{\pi}{4}} x\cdot(\cos(x)-\sin(x))\,dx}{\int_0^{\frac{\pi}{4}} (\cos(x)-\sin(x))\,dx} =$$

$$\frac{-1+\frac{1}{4}\sqrt{2}\,\pi}{\sqrt{2}-1}$$

0.2673034979

$$\bar{y} = \frac{\int_a^b y\, g(y)\,dy}{\int_a^b g(y)\,dy}$$

$$= \frac{\int_0^{\frac{1}{\sqrt{2}}} y\cdot\arcsin(y)\,dy + \int_{\frac{1}{\sqrt{2}}}^{1} y\cdot\arccos(y)\,dy}{\int_0^{\frac{1}{\sqrt{2}}} \arcsin(y)\,dy + \int_{\frac{1}{\sqrt{2}}}^{1} \arccos(y)\,dy}$$

$$= \frac{\frac{1}{4}}{\sqrt{2}-1} \approx 0.6035533912$$