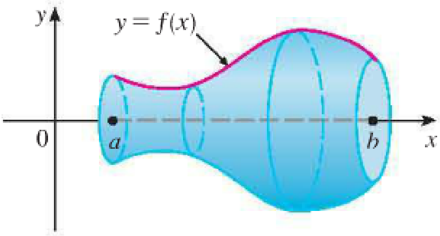
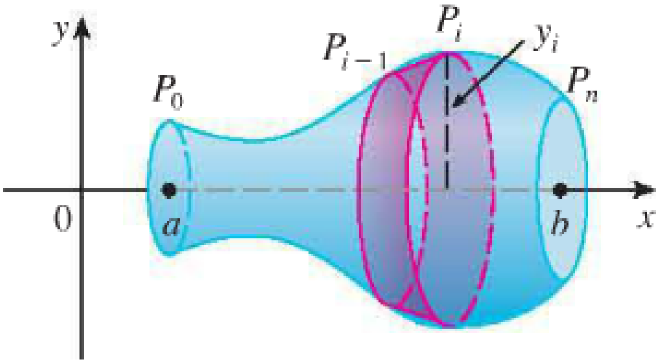
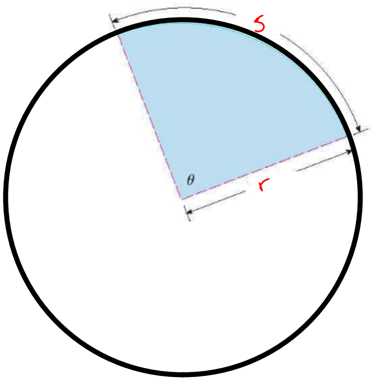
Section 8.2
Surface Area of Solids of Revolution

S 8.2 #s 1, 5, 6, 13, 14, 21, 22, 25, 26







Circumference of Circle $= 2\pi r$ $(\theta = 2\pi)$

Area of a Circle $= \pi r^2 = \frac{1}{2}(2\pi) r^2$ $(\theta = 2\pi)$

In general,

Arc Length $= S = r\theta$

Area of sector $= A = \frac{1}{2} r^2 \theta$ or $\frac{1}{2}\theta r^2$

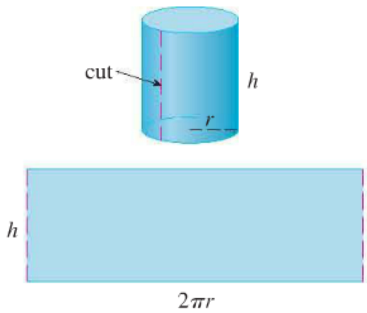
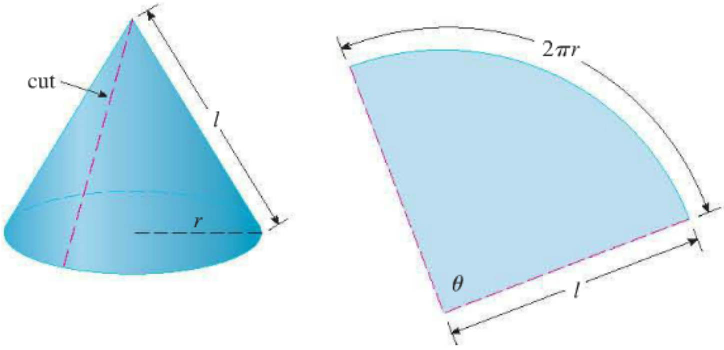


FIGURE 1

$A = 2\pi r h$



$A = \frac{1}{2} l^2 \theta$

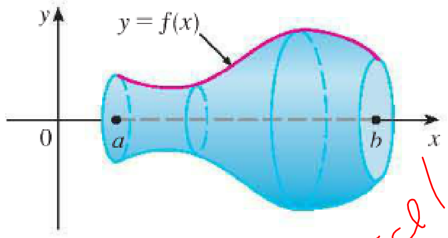
$= \frac{1}{2} l^2 \left(\frac{2\pi r}{l}\right)$

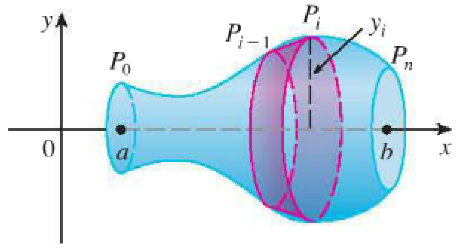
$= \frac{1}{2} 2\pi r l$

$= \pi r l$

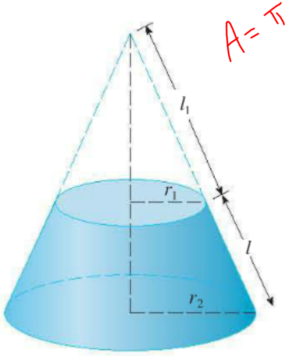
$S = l\theta = 2\pi r$

$= \theta = \frac{2\pi r}{l}$





$A = \pi r \ell$!



$$\frac{\ell_1+\ell}{r_2} = \frac{\ell_1}{r_1}$$

$(\ell_1+\ell)r_1 = \ell_1 r_2$

$\ell_1 r_1 + \ell r_1 = \ell_1 r_2$

$\ell r_1 = \ell_1 r_2 - \ell_1 r_1 = \ell_1(r_2 - r_1)$

Surface Area of Frustum.

= Area of Big cone − Area of smaller cone

$= \pi r_2(\ell+\ell_1) - \pi r_1 \ell_1$

$= \pi r_2 \ell + \pi r_2 \ell_1 - \pi r_1 \ell_1$

$= \pi r_2 \ell + \pi \ell_1 (r_2 - r_1)$

$= \pi r_2 \ell + \pi \ell r_1$

$= \pi \ell (r_1 + r_2)$

$= 2\pi \ell \left(\frac{r_1+r_2}{2}\right)$

$= 2\pi \ell r$, where r = avg radius.

$= 2\pi r \ell$ $\qquad \ell$ = arc length.

$= 2\pi f(x) \cdot ds$ $\qquad$ arc length $= \int_a^b \sqrt{1+(f'(x))^2}\,dx$

Differential of arc length $= ds = \sqrt{1+(f'(x))^2}\,dx$

Revolving around the x-axis:

$y = f(x) \Rightarrow$

$$\boxed{4} \quad S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$\boxed{5} \quad S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$x = g(y) \Rightarrow$

$$\boxed{6} \quad S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

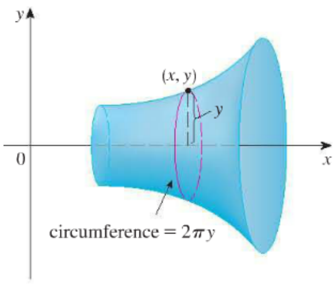
$(g'(y))^2$

$$\boxed{7} \quad S = \int 2\pi y\,ds$$

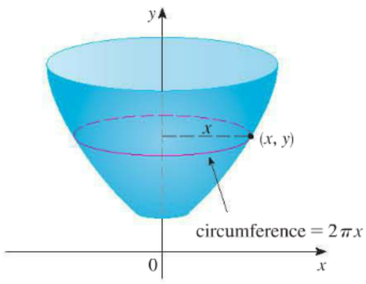
Revolving around the y-axis:

$$\boxed{8} \quad S = \int 2\pi x\,ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$



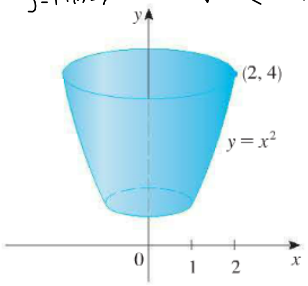
(a) Rotation about x-axis: $S = \int 2\pi y\, ds$



(b) Rotation about y-axis: $S = \int 2\pi x\, ds$

EXAMPLE 2 The arc of the parabola $y = x^2$ from (1, 1) to (2, 4) is rotated about the y-axis. Find the area of the resulting surface.

$y=f(x) \Rightarrow ds = \sqrt{1+(f'(x))^2}\,dx$



$$2\pi\int_1^2 x\,ds = 2\pi\int_1^2 x\sqrt{1+(2x)^2}\,dx$$

OR

$x=g(y)$: $y=x^2 \Rightarrow x=\pm\sqrt{y} \Rightarrow x=\sqrt{y} = g(y)$

$ds=\sqrt{1+(g'(y))^2}\,dy$

$$2\pi\int_1^4 x\,ds = 2\pi\int_1^4 \sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

$$2\cdot\pi\int_1^2 x\cdot\sqrt{1+(2\cdot x)^2}\,dx$$

$$2\pi\left(-\frac{5}{12}\sqrt{5}+\frac{17}{12}\sqrt{17}\right)$$

$evalf(\%)$

$$30.84648972$$

$$2\cdot\pi\int_1^4 \sqrt{y}\cdot\sqrt{1+\left(\frac{1}{2\cdot\sqrt{y}}\right)^2}\,dy$$

$$2\pi\left(-\frac{5}{12}\sqrt{5}+\frac{17}{12}\sqrt{17}\right)$$

$evalf(\%)$

$$30.84648972$$

1–4

(a) Set up an integral for the area of the surface obtained by rotating the curve about (i) the x-axis and (ii) the y-axis.

(b) Use the numerical integration capability of your calculator to evaluate the surface areas correct to four decimal places.

1. $y = \tan x, \quad 0 \leqslant x \leqslant \pi/3$

a (i) $2\pi \int y \, ds$

(ii) $2\pi \int x \, ds$

b Wolfram Alpha or any other CAS is OK, as well as graphing calculator

5–12 Find the exact area of the surface obtained by rotating the curve about the x-axis.

5. $y = x^3, \quad 0 \leqslant x \leqslant 2$

#22?

$y' = 3x^2$

$(y')^2 = 9x^4 = (3x^2)^2 = 9(x^2)^2$

$2\pi \int y\,ds = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx$

$u = x^2$

$du = 2x$

$x^3 = u \cdot x$

$= 2\pi \int_0^4 u \cdot x \sqrt{1+9u^2}\, dx$

$= 2\pi \cdot \frac{1}{2} \int_0^4 u\sqrt{1+9u^2} \cdot 2x\,dx$

$= \pi \int_0^4 u\sqrt{1+9u^2}\, du$

$v = 1+9u^2$

$dv = 18u\, du$

$= \frac{\pi}{18} \int_{u=0}^{u=4} \sqrt{1+9u^2} \cdot 18u\, du$

$= \frac{\pi}{18} \int_{u=0}^{u=4} \sqrt{v}\, dv$

$= \frac{\pi}{18} \cdot \frac{2}{3} v^{\frac{3}{2}} \Big]_{u=0}^{u=4}$

$= \frac{\pi}{27}\left[(1+9u^2)^{3/2}\right]_0^4$

$= \frac{\pi}{27}\left[145^{\frac{3}{2}} - 1\right]$

$\approx$

$\begin{array}{r} {}^{5}16 \\ 9 \\ \hline 144 \end{array}$

$$2\cdot\pi\cdot\int_0^2 x^3\cdot\sqrt{1+9\cdot x^4}\,dx$$

$$-\frac{1}{36}\sqrt{\pi}\left(-\frac{580}{3}\sqrt{\pi}\sqrt{145}+\frac{4}{3}\sqrt{\pi}\right)$$

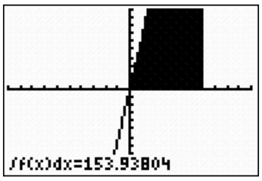
$evalf(\%)$

$$203.0436016$$

$$evalf\left(\frac{\pi}{27}\cdot\left(145^{\frac{3}{2}}-1\right)\right)$$

$$203.0436016$$

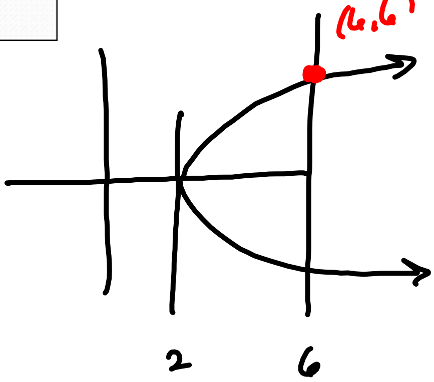
6. $9x = y^2 + 18, \quad 2 \leq x \leq 6$



$9x = y^2 + 18$ about x-axis, $2 \leq x \leq 6$

$x = \frac{1}{9}y^2 + 2$

$2\pi \int y\, ds \qquad \frac{dx}{dy} = \frac{2}{9}y$



$= 2\pi \int_0^6 y \sqrt{1 + \frac{4}{81}y^2}\, dy$

Q: #6

$y^2 + 18 = 9x$

$y^2 = 9x - 18$

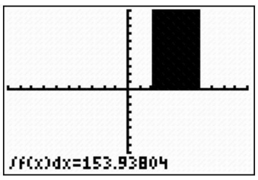
$y = \pm\sqrt{9x-18} = \pm 3\sqrt{x-2}$

$y = 3\sqrt{x-2}$

$= 3(x-2)^{\frac{1}{2}}$

$y' = \frac{3}{2}(x-2)^{-\frac{1}{2}}$

$2\pi \int_2^6 3\sqrt{x-2}\sqrt{1 + \frac{9}{4}(x-2)^{-1}}\, dx$



13–16 The given curve is rotated about the y-axis. Find the area of the resulting surface.

$(1, 1)$, $(2^3, 2)$

13. $y = \sqrt[3]{x}$, $1 \leqslant y \leqslant 2$

$y = \sqrt[3]{x}$

$y^3 = x$

$f'(x) = y' = \frac{1}{3}x^{-\frac{2}{3}}$

$2\pi \int x \, ds$

$ds = \sqrt{1 + \left(\frac{1}{3}x^{-2/3}\right)^2} \, dx$

$= 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9}x^{-\frac{4}{3}}} \, dx$

Let $u = \frac{1}{3}x^{-\frac{2}{3}}$

$u = \frac{1}{3}x^{-\frac{2}{3}}$

$3x^{\frac{2}{3}} = \frac{1}{u}$

$x^{\frac{2}{3}} = \frac{1}{3u}$

$x = \left(\frac{1}{3u}\right)^{\frac{3}{2}} = \frac{1}{3}u^{-\frac{3}{2}}$

$dx = -\frac{3}{2}u^{-\frac{5}{2}} du$

$= 2\pi \int_{x=1}^{x=8} \left(\frac{1}{3u}\right)^{\frac{3}{2}} \sqrt{1 + u^2} \left(-\frac{1}{2}u^{-\frac{5}{2}} du\right)$

$= 2\pi \left(\frac{1}{3}\right)^{\frac{3}{2}} \left(-\frac{3}{2}\right) \int_{x=1}^{x=8} u^{-4} \sqrt{1 + u^2} \, du$

13. $y = \sqrt[3]{x}, \quad 1 \leqslant y \leqslant 2$ about y-axis

$x = y^3, \; 1 \le y \le 2$

$$2\pi \int_1^2 y^3 \sqrt{1 + (3y^2)^2}\, dy = 2\pi \int_{y=1}^{y=2} \left(\frac{u}{3}\right)^3 \sqrt{1+u^2}\, \frac{du}{6y}$$

$u = 3y^2$

$\frac{u}{3} = y^2$

$du = 6y\, dy$

$\Rightarrow dy = \frac{du}{6y}$

$y^3 = \left(\frac{u}{3}\right)^3$

$$= \frac{2\pi}{27} \cdot \frac{1}{6} \int_{y=1}^{y=2} u^3 \sqrt{1+u^2}\, \frac{du}{\left(\frac{u}{3}\right)}$$

$$= \frac{\pi}{81} \cdot 3 \int_{y=1}^{y=2} u^2 \sqrt{1+u^2}\, du$$

#22

$$= \frac{\pi}{27} \left[\frac{u}{8}(1 + 2u^2) - \frac{1}{8} \ln\left(u + \sqrt{1+u^2}\right) \right]_{y=1}^{y=2}$$

$$= \frac{\pi}{27} \left[\frac{3y^2}{8}\left(1 + 2(3y^2)^2\right) - \frac{1}{8} \ln\left(3y^2 + \sqrt{1 + (3y^2)^2}\right) \right]_1^2$$

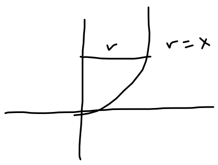
$$= \frac{\pi}{8 \cdot 27} \Big[3 \cdot 4\left(1 + 2(3 \cdot 4)^2\right) - \ln\left(3(2)^2 + \sqrt{1 + 12^2}\right.$$
$$\left. - \left(3(1 + 2(3)^2 - \ln(3 + \sqrt{1 + 3^2})\right) \right]$$

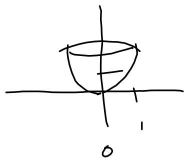
$$= \frac{\pi}{(8)(27)} \left[12(1 + 288) - \ln\left(12 + \sqrt{145}\right) - \left(3(19) + \ln(3 + \sqrt{10})\right) \right]$$

$$= \frac{\pi}{8(27)} \left[12(289) - \ln(12 + \sqrt{145} - 57 + \ln(3 + \sqrt{10}) \right]$$

14. $y = 1 - x^2, \quad 0 \leqslant x \leqslant 1$

about y-axis





$y' = -2x$

$(y')^2 = 4x^2$

$2\pi \int_0^1 x\, ds$

$= 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx = 2\pi \int_1^5 \sqrt{u} \cdot x \cdot \frac{du}{8x}$

$u = 1+4x^2$

$du = 8x\, dx$

$dx = \frac{du}{8x}$

$x=1 \Rightarrow u = 1+4=5$

$x = 0 \Rightarrow u = 1$

$= \frac{1}{4}\pi \int_1^5 u^{\frac{1}{2}} du$

$= \frac{1}{4}\pi \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5$

$= \frac{1}{4}\pi \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^5$

$= \frac{\pi}{6}\left[5^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \frac{\pi}{6}\left[5^{\frac{3}{2}} - 1\right]$

$= \frac{\pi}{6} \cdot 5\sqrt{5} - \frac{\pi}{6}$

$(5\sqrt{5} - 1)\frac{\pi}{6}$

$\int u^{-1} du$

$= \ln|u| + C$

$\sqrt[2]{5^3}$

$= \sqrt[2]{5 \cdot 5 \cdot 5}$

$5\sqrt{5}$

$5^{\frac{3}{2}} = (5^3)^{\frac{1}{2}}$

$= (5^{\frac{1}{2}})^3$

$= 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$

$= 5^1 \cdot 5^{\frac{1}{2}} = 5\sqrt{5}$

$=$

#14:

$2 \cdot \pi \int_0^1 x \cdot \sqrt{1 + (-2 \cdot x)^2}\, dx = -\frac{1}{8}\sqrt{\pi}\left(-\frac{20}{3}\sqrt{\pi}\sqrt{5} + \frac{4}{3}\sqrt{\pi}\right)$

$expand(\%) = \frac{5}{6}\pi\sqrt{5} - \frac{1}{6}\pi$

$evalf(\%) \approx 5.330413499$

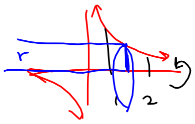
21–22 Use either a CAS or a table of integrals to find the exact area of the surface obtained by rotating the given curve about the x-axis.

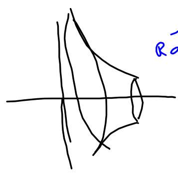
21. $y = 1/x, \quad 1 \leqslant x \leqslant 2$

x-axis

$2\pi \int y \, ds$

Radius. $= f(x) = y$





$y = \frac{1}{x} = x^{-1} \implies$

$y' = -x^{-2}$

$(-x^{-2})^2 = x^{-4}$

$2\pi \int_1^2 \frac{1}{x} \sqrt{1 + x^{-4}} \, dx$

$u = x^{-2}$

$du = -2x^{-3} \, dx$

New p.

$2 \cdot \mathrm{Pi} \cdot \int_1^2 \frac{1}{x} \cdot \sqrt{1 + x^{-4}} \, dx \qquad 2\pi \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}+1) - \frac{1}{8}\sqrt{17} + \frac{1}{2}\ln(4+\sqrt{17}) \right)$

evalf(%) 5.016420118

S 8.2 #s 1, 5, 6, 13, 14, 21, 22, 25, 26

22. $y = \sqrt{x^2+1}, \quad 0 \leqslant x \leqslant 3$, x axis

$$2\pi \int y \, ds$$

$$y = (x^2+1)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

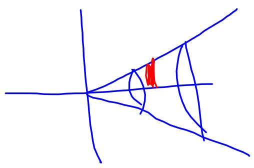
$$= \frac{x}{\sqrt{x^2+1}}$$

$$(y')^2 = \frac{x^2}{x^2+1}$$

$$2\pi \int_a^b y \, ds = 2\pi \int_0^3 \sqrt{x^2+1} \sqrt{1 + \frac{x^2}{x^2+1}} \, dx$$

$$\sqrt{\frac{x^2+1+x^2}{x^2+1}}$$

$$\frac{\sqrt{2x^2+1}}{\sqrt{x^2+1}}$$

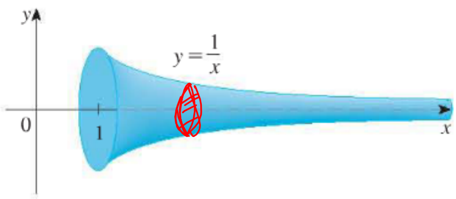


$$= 2\pi \int_0^3 \sqrt{x^2+1} \, \frac{\sqrt{2x^2+1}}{\sqrt{x^2+1}} \, dx$$

$$= 2\pi \int_0^3 \sqrt{2x^2+1} \, dx$$

$$\approx 45.86206616$$

25. If the region $\mathcal{R} = \{(x, y) \mid x \geq 1,\ 0 \leq y \leq 1/x\}$ is rotated about the x-axis, the volume of the resulting solid is finite (see Exercise 63 in Section 7.8). Show that the surface area is infinite. (The surface is shown in the figure and is known as **Gabriel's horn**.)



$$2\pi \int_1^{\infty} y \, ds$$

$$\lim_{t \to \infty} 2\pi \int_1^{t} \frac{1}{x} \sqrt{1 + x^{-4}}\, dx = \infty \,!$$

$y = x^{-1}$

$y' = -x^{-2}$

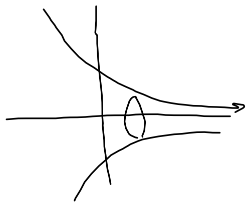
$(y')^2 = x^{-4}$

Vol

$$\pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx = \pi \int_1^{\infty} x^{-2}\, dx = \lim_{t \to \infty} \left[-x^{-1}\right]_1^{t} = \pi$$

26. If the infinite curve $y = e^{-x}$, $x \geq 0$, is rotated about the x-axis, find the area of the resulting surface.

$$2\pi \int y \, ds = 2\pi \int_0^{\infty} e^{-x} \sqrt{1 + e^{-2x}} \, dx$$



$y = e^{-x}$

$y' = -e^{-x}$

$(y')^2 = e^{-2x}$

$u = e^{-x}$

$du = -e^{-x} dx$

$dx = -\frac{du}{e^{-x}} = -e^{x} du$

$$2\pi \int_0^{\infty} e^{-x} \sqrt{1+u^2} \cdot (-e^{x} du)$$

$$= -2\pi \int_1^{0} \sqrt{1+u^2} \, du$$

$u = e^{-x}$

$x = 0 \Longrightarrow u = e^{-0} = 1$

$x = \infty \Longrightarrow u = e^{-\infty} = 0$