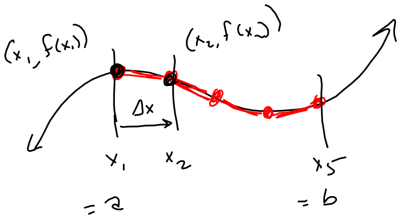
Section 8.1 - Arc Length

S 8.1 #s 2, 3, 5, 7, 9, 18, 39



$$\frac{\Delta y}{\Delta x}$$

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2\left[1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right]}$$

$$= \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

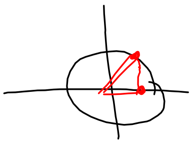
$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

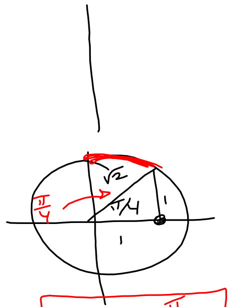
So the approximate arc length

$$\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$\xrightarrow[\Delta x \to 0]{n \to \infty} \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

2. Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leqslant x \leqslant 1$. Check your answer by noting that the curve is part of a circle.





$$f(x) = (2-x^2)^{\frac{1}{2}}$$

$$f'(x) = \tfrac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x)$$

$$= -\frac{x}{(2-x^2)^{\frac{1}{2}}}$$

$$\frac{2-x^2}{2-x^2} + \frac{x^2}{2-x^2}$$

$$= \frac{2}{2-x^2}$$

#16

$$\boxed{r\theta = \sqrt{2} \cdot \frac{\pi}{4}}$$

Arc length formula for radians!

$$(f'(x))^2 = \frac{x^2}{2-x^2}$$

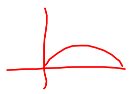
$$\int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + \frac{x^2}{2-x^2}}\, dx = \int_0^1 \sqrt{\frac{2}{2-x^2}}\, dx$$

$$= \int_0^1 \frac{\sqrt{2}\, dx}{\sqrt{2-x^2}} = \sqrt{2}\left[\arcsin\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$$

$$= \sqrt{2}\left[\arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin(0)\right]$$

$$= \sqrt{2} \cdot \frac{\pi}{4} = \boxed{\frac{\sqrt{2}\,\pi}{4}}$$

3–6 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

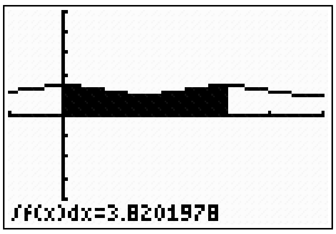


3. $y = \sin x, \quad 0 \leqslant x \leqslant \pi$

This requires a graphing calculator or Computer Algebra App. Please share what you're using, if you've found a good phone app. Worst case, you can use a school computer with Excel, and do approximate integrals that are very close, using Chapter 7 techniques. I can show you how.

These arc length integrals are generally tough to evaluate. The book has some exercises cooked up to come out nice, later in the problem set. On a test, it'd stop right at "write the integral," unless I had one 'specially cooked-up to come out nice, in short time.

$$\int_0^{\pi} \sqrt{1 + \cos^2 x}\; dx \approx 3.8201978$$



5. $x = \sqrt{y} - y, \quad 1 \leqslant y \leqslant 4$

$$\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$g(y) = x = y^{\frac{1}{2}} - y \implies$$

$$x' = \frac{1}{2}y^{-\frac{1}{2}} - 1$$

$$\left(\frac{dx}{dy}\right)^2 = (x')^2 = \left(\frac{1}{2}y^{-\frac{1}{2}}\right)^2 - 2\left(\frac{1}{2}\right)y^{-\frac{1}{2}} + 1$$

$$= \frac{1}{4}y^{-1} - y^{-\frac{1}{2}} + 1$$

$$\int_1^4 \sqrt{1 + \frac{1}{4}y^{-1} - y^{-\frac{1}{2}} + 1}\, dy$$

$$= \int_1^4 \sqrt{2 + \frac{1}{4y} - \frac{1}{\sqrt{y}}}\, dy = \int_1^4 \sqrt{\frac{8y\sqrt{y} + \sqrt{y} - 4y}{4y\sqrt{y}}}\, dy \approx 3.609454496$$

7–18 Find the exact length of the curve.

7. $y = 1 + 6x^{3/2}$, $0 \leqslant x \leqslant 1$

$y' = 9x^{\frac{1}{2}}$

$(y')^2 = 81x$

$$\int_0^1 \sqrt{1+81x}\,dx = \frac{1}{81}\int_0^1 \sqrt{81x+1}\cdot 81\,dx = \frac{1}{81}\int_{x=0}^{x=1} u^{\frac{1}{2}}\,du = \frac{1}{81}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big]_{x=0}^{x=1}$$

$u = 81x+1$

$du = 81\,dx$

$$= \frac{2}{243}\left[(81x+1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{243}\left[82^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \boxed{\frac{2}{243}\left[82^{\frac{3}{2}} - 1\right]}$$

9. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \leqslant x \leqslant 2$ $\qquad \frac{1}{3}x^3 + \frac{1}{4}x^{-1}$

$y' = x^2 - \frac{1}{4}x^{-2}$

$(y')^2 = x^4 - 2(x^2)(\frac{1}{4}x^{-2}) + \frac{1}{16}x^{-4}$

$= x^4 - \frac{1}{2} + \frac{1}{16}x^{-4}$

$\int_a^b \sqrt{1+(y')^2}\,dx = \int_1^2 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16}x^{-4}}\,dx = \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16}x^{-4}}\,dx$

$= \int_1^2 \sqrt{(x^2 + \frac{1}{4}x^{-2})^2}\,dx = \int_1^2 |x^2 + \frac{1}{4}x^{-2}|\,dx$

$= \int_1^2 (x^2 + \frac{1}{4}x^{-2})\,dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^{-1}\right]_1^2 = \frac{1}{3}\cdot 8 - \frac{1}{4}\cdot\frac{1}{2} - \left[\frac{1}{3} - \frac{1}{4}\right]$

$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$

$= \dfrac{64 - 3 - 8 + 6}{24} = \boxed{\dfrac{59}{24}}$

18. $y = 1 - e^{-x}, \quad 0 \leqslant x \leqslant 2$

1st Attempt

Maybe we can iron this out in class? I'm failing this one & must move on.

$y' = e^{-x}$

$(y')^2 = (e^{-x})^2 = e^{-2x}$

$$\int_0^2 \sqrt{1+e^{-2x}}\,dx = \left[-\frac{\sqrt{1+e^{2x}}}{e^x} + \ln\left(e^x + \sqrt{1+e^{2x}}\right)\right]_0^2$$

$u = e^{-x}$

$$= -\frac{\sqrt{e^4+1}}{e^2} + \ln\left(e^2+\sqrt{e^4+1}\right) - \left(-\frac{\sqrt{2}}{1} + \ln(1+\sqrt{2})\right)$$

$$= -\frac{\sqrt{e^4+1}}{e^2} + \ln\left(e^2+\sqrt{e^4+1}\right) + \sqrt{2} - \ln(\sqrt{2}+1)$$

$$\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}-1) - \frac{1}{2}\ln(1+\sqrt{2}) - \sqrt{1+e^{-4}} - \frac{1}{2}\ln\left(\sqrt{1+e^{-4}}-1\right) + \frac{1}{2}\ln\left(1+\sqrt{1+e^{-4}}\right)$$

$$1 + e^{-2x} = 1 + \frac{1}{e^{2x}} = \frac{e^{2x}+1}{e^{2x}}$$

$$\sqrt{\frac{e^{2x}+1}{e^{2x}}}\,dx$$

$$\frac{\sqrt{e^{2x}+1}}{e^x}\,dx$$

$u = e^x$

$du = e^x\,dx \Rightarrow \frac{du}{e^x} = dx = \frac{du}{u}$

$$= \int \frac{\sqrt{u^2+1}}{u}\cdot\frac{du}{u}$$

$$\int \frac{\sqrt{u^2+1}}{u^2}\,du$$

#24 $a=1$

$$-\frac{\sqrt{1+u^2}}{u} + \ln\left(u+\sqrt{1+u^2}\right) + C$$

18. $y = 1 - e^{-x}, \quad 0 \leqslant x \leqslant 2$

2^{nd} Attempt
Looks better, but struggling to get agreement with other sources

$y' = e^{-x}$

$(y')^2 = e^{-2x}$

$\int_0^2 \sqrt{1+e^{-2x}}\,dx = -\int_{0=x}^{x=2} \sqrt{1+u^2} \cdot u\,du = -\frac{1}{2}\int_{x=0}^{x=2} \sqrt{u^2+1} \cdot 2u\,du$

$u = e^{-x}$

$du = -e^{-x}\,dx$

$dx = -e^{x}du = -u\,du$

$v = u^2+1$

$dv = 2u\,du$

$= -\frac{1}{2}\int_{x=0}^{x=2} v^{\frac{1}{2}}\,dv = -\frac{1}{2}\frac{v^{\frac{3}{2}}}{\frac{3}{2}}\Big]_{x=0}^{x=2}$

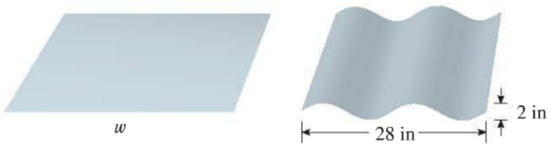
$= -\frac{1}{3}v^{\frac{3}{2}}\Big]_{x=0}^{x=2} = -\frac{1}{3}(u^2+1)^{\frac{3}{2}}\Big]_{x=0}^{x=2}$

$= -\frac{1}{3}\Big[(e^{-2x}+1)^{\frac{3}{2}}\Big]_{x=0}^{x=2}$

$= -\frac{1}{3}\Big[(e^{-4}+1)^{\frac{3}{2}} - (1+1)^{\frac{3}{2}}\Big]$

$= -\frac{1}{3}\Big[(e^{-4}+1)^{\frac{3}{2}} - 2^{\frac{3}{2}}\Big]$

39. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = \sin(\pi x/7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel. (Use your calculator to evaluate the integral correct to four significant digits.)



$\sin\left(\frac{\pi x}{7}\right) = \sin\left(\frac{\pi}{7}x\right)$

$\frac{\pi}{7}x = 2\pi \Rightarrow x = 14$ is its period, and we're looking @ $x=28$ is 2 periods' worth

$f(x) = \sin\left(\frac{\pi}{7}x\right)$

$f'(x) = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$

$(f'(x))^2 = \frac{\pi^2}{49}\cos^2\left(\frac{\pi}{7}x\right)$

$$w = \int_0^{28} \sqrt{1 + \frac{\pi^2}{49}\cos^2\left(\frac{\pi}{7}x\right)}\,dx$$

≈ 29.36072662, by MAPLE