§7.8 #s 5, 10, 14, 20, 25, 33, 50, 54, 56

§7.8 Think back to the 1^{st} fundamental theorem, when we TRIED to think of

Improper Integrals

$$G(x) = \int_a^x f(t)\,dt$$ as a function of x, and even differentiate it wrt x. we're going to use this idea, when facing things like

$$\int_1^\infty \frac{1}{t^3}\,dt$$, by re-writing it as

$$\lim_{x\to\infty} \int_1^x \frac{1}{t^3}\,dt = \lim_{x\to\infty} \left[\frac{t^{-2}}{-2}\right]_1^x = \lim_{x\to\infty} \left[-\frac{1}{2x^2} - \left(-\frac{1}{2(1)^2}\right)\right]$$

$$\int t^{-3}\,dt = \frac{t^{-2}}{-2} + C$$

$$= \lim_{x\to\infty} \left[-\frac{1}{2x^2} + \frac{1}{2}\right] = \frac{1}{2} \;!$$

p-test

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Series : $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^p} + \cdots$ converges.

5–40 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \int_1^\infty \frac{1}{u^{3/2}}\,du = \lim_{t\to\infty} \int_1^t \frac{1}{u^{3/2}}\,du = \lim_{t\to\infty} \left[-\frac{1}{\frac{1}{2}} \cdot u^{-\frac{1}{2}}\right]_1^t$

$u = x-2$
$du = dx$

$x = 3 \rightarrow u = 1$

$= \lim_{t\to\infty} \left[-2t^{-\frac{1}{2}} - (-2(1)^{-\frac{1}{2}}\right]$

$= 0 + 2 = \boxed{2}$

10. $\int_{-\infty}^{0} 2^r\,dr = \lim_{t\to-\infty} \int_t^0 2^r\,dr$

$\int 2^x\,dx = \frac{1}{\ln 2} 2^x + C$

$\stackrel{?}{=} (\ln 2)\cdot 2^x + C$

$2^r = e^{\ln(2^r)} = e^{r\ln(2)} = e^{\ln(2)r}$

$= \lim_{t\to-\infty} \frac{1}{\ln(2)}\cdot 2^r \Big]_t^0 = \lim_{t\to-\infty}\left[\frac{1}{\ln(2)}\cdot 2^0 - \frac{1}{\ln(2)}\cdot 2^t\right] = \frac{1}{\ln(2)}$

0

$\frac{1}{\ln(2)}\int e^{\ln(2)r}\cdot \ln(2)\,dr$

$= \frac{\cancel{\ln(2)}\,\cancel{2}^r}{\ln(2)} + C$ $\ln(2^r)$

$= \frac{1}{\ln(2)} e^{\ln(2)r} + C$

$\int e^{\ln(2)r}\,dr = \int e^u \frac{du}{\ln(2)}$

$u = \ln(2)r$

$du = \ln(2)\,dr$

$\frac{du}{\ln(2)} = dr$

$= \frac{1}{\ln(2)}\int e^u\,du$

$= \frac{1}{\ln(2)} e^{\ln(2)r} + C$

$= \frac{1}{\ln(2)}\cdot 2^r + C$

Sloppy handling of the limits

14. $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \int_{1=x}^{\infty} \frac{e^u}{\sqrt{x}} \cdot -2\sqrt{x}\,du = \int_{-1}^{-\infty} -2e^u\,du = \lim_{t\to-\infty} -2\int_{-1}^{t} e^u\,du$

Find $\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$, first

$u = -\sqrt{x} = -x^{\frac{1}{2}}$, then

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx = -\frac{1}{2\sqrt{x}}\,dx$

$-2\sqrt{x}\,du = dx$

$\lim_{x\to\infty} = \lim_{-\sqrt{x}\to-\infty}$

$= \lim_{t\to\infty} -2e^u\Big]_{-1}^{t}$

$= \lim_{t\to-\infty} \left(-2e^t - (-2e^{-1})\right)$

$= \boxed{\frac{2}{e}}$

Cleaner handling of the limits

14. $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} -2e^{-\sqrt{x}}\Big]_1^t = \lim_{t\to\infty} -2\Big[\underbrace{e^{-\sqrt{t}}}_{0} - e^{-\sqrt{1}}\Big] = 2e^{-1}$ OR $\dfrac{2}{e}$

Find $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$, first $\displaystyle = \int \frac{e^{u}}{\sqrt{x}} \cdot -2\sqrt{x}\,du = -2\int e^{u}\,du$

$u = -\sqrt{x} = -x^{\frac{1}{2}}$, then

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx = -\dfrac{1}{2\sqrt{x}}\,dx$

$-2\sqrt{x}\,du = dx$

$= -2e^{u} + C$

$= -2e^{-\sqrt{x}} + C$

20. $\int_2^\infty ye^{-3y}\,dy = \lim_{t\to\infty}\left[-\frac{1}{3}ye^{-3y} - \frac{1}{9}e^{-3y}\right]_2^t = \lim_{t\to\infty}\left[-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} - \left(-\frac{1}{3}(2)e^{-6} - \frac{1}{9}e^{-6}\right)\right]$

$u = y \qquad dv = e^{-3y}dy$

$du = dy \qquad v = -\frac{1}{3}e^{-3y}$

$= \frac{2}{3}e^{-6} + \frac{1}{9}e^{-6} = \left(\frac{6}{9} + \frac{1}{9}\right)e^{-6}$

$= \boxed{\frac{7}{9}e^{-6}}$

$\int ye^{-3y}\,du = uv - \int v\,du = -\frac{1}{3}y\,e^{-3y} - \int -\frac{1}{3}e^{-3y}dy$

$= -\frac{1}{3}ye^{-3y} + \frac{1}{3}\cdot\left(-\frac{1}{3}e^{-3y}\right) + C$

$= -\frac{1}{3}ye^{-3y} - \frac{1}{9}e^{-3y} + C$

25. $\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} -\frac{1}{2}\left[(\ln x)^{-2}\right]_e^t = \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{\ln(t)^2} - \frac{1}{\ln(e)^2}\right]$

$= \boxed{\frac{1}{2}}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int (\ln x)^{-3} \cdot \frac{1}{x}\,dx = -\frac{1}{2}(\ln x)^{-2} + C$

Another type of p-test, for functions that have ∞ discontinuities away from $x \to \pm\infty$

$\int_0^1 \frac{1}{x^p}\,dx$ converges for all $p < 1$.

The other p-test

$\int_1^\infty \frac{1}{x^p}\,dx$ converges $\forall\, p > 1$

33. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

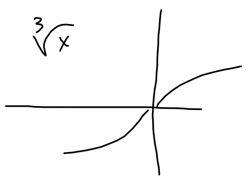
$u = x-1$
$du = dx$

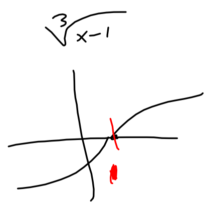
$$\int (x-1)^{-\frac{1}{3}}\,dx = \int u^{-\frac{1}{3}}\,du = \frac{3}{2}u^{\frac{2}{3}} + C = \frac{3}{2}\sqrt[3]{(x-1)^2} + C$$

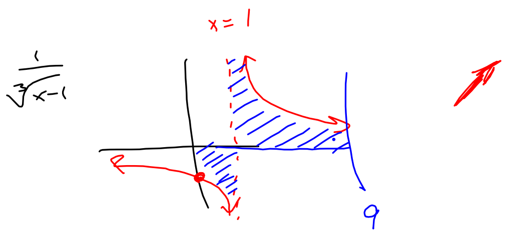
33. $$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 f(x)\,dx + \int_1^9 f(x)\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{t\to 1^+}\int_t^9 \frac{dx}{\sqrt[3]{x-1}}$$

$$= \frac{3}{2}\left[\lim_{t\to 1^-}\left[\sqrt[3]{(x-1)^2}\right]_0^t + \left[\lim_{t\to 1^+}\sqrt[3]{(x-1)^2}\right]_t^9\right] = -\frac{3}{2}(1) + \frac{3}{2}(4)$$

$$= \frac{9}{2}\ !$$







33. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_{-1}^{8} \frac{1}{u^{1/3}}\,du = \int_{-1}^{8} u^{-\frac{1}{3}}\,du = \int_{-1}^{1} u^{-\frac{1}{3}}\,du + \int_{1}^{8} u^{-\frac{1}{3}}\,du$

$\displaystyle\int_{-1}^{1} u^{-\frac{1}{3}}\,du \to 0$

$u = x-1$

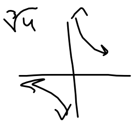
$x=0 \Rightarrow u=-1$

$x=1 \Rightarrow u=0$

$x=2 \Rightarrow u=1$

$x=9 \Rightarrow u=8$

$\frac{1}{\sqrt[3]{u}}$



$u^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{u}}$ is odd function

$\frac{1}{\sqrt[3]{8}} = \frac{1}{2}$, $\frac{1}{\sqrt[3]{-8}} = -\frac{1}{2}$

It boils down to

$\displaystyle\int_1^8 u^{-\frac{1}{3}}\,du = \frac{3}{2}u^{\frac{2}{3}}\Big]_1^8 = \frac{3}{2}(4) - \frac{3}{2}(1)$

$= \frac{12-3}{2} = \frac{9}{2} = 4.5$

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Lebesgue Dominated Convergence Theorem

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

50. $\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$ Diverges, by comparison to the smaller $\int \frac{2}{x}\,dx$

$e^{-x} > 0 \implies \frac{2+e^{-x}}{x} > \frac{2}{x}$ would be cool, if

if $\int \frac{2}{x}$ converged

& $\int_1^\infty \frac{2}{x}\,dx$ diverges, b/c p-test x^1

$\int \frac{2+e^{-x}}{x}\,dx \geq \int \frac{2}{x}\,dx$

54. $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ A type 2

$= \lim_{b \to 0^+} \int_b^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

$\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ & $\int_0^\pi x^{-\frac{1}{2}}\,dx$ converges.

Converges

55. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

56. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

by the same method as in Exercise 55.

$$= \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}}$$

$$= \lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty} \int_3^t \frac{dx}{x\sqrt{x^2-4}}$$

#18 $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{u}{a}\right) + C$

$u = x,\ a = 2$

$$= \lim_{t\to 2^+} \frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\Big]_t^3 + \lim_{t\to\infty} \frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\Big]_3^t$$

$$= \lim_{t\to 2^+}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right) - \frac{1}{2}\operatorname{arcsec}\left(\frac{t}{2}\right)\right] + \lim_{t\to\infty}\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{t}{2}\right) - \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}\right)\right]$$

$$= -\frac{1}{2}\operatorname{arcsec}\left(\frac{2}{2}\right) + \frac{1}{2}\cdot\frac{\pi}{2} = -\frac{1}{2}\cdot 0 + \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$