

FACT: There is NO elementary function that is the anti-derivative of $y = e^{-x^2}$!

That's why we do it, digitally, numerically.

I will use a spreadsheet to make the work go semi-quickly. There's not much theory in this section. But Simpson's Rule deserves discussion.

We approximate $f(x)$ by fitting a parabola between 3 points.

3 (non-collinear) points define a parabola $f(x) = Ax^2 + Bx + C$

$(x_1, y_1), (x_2, y_2), (x_3, y_3)$ give us a system in

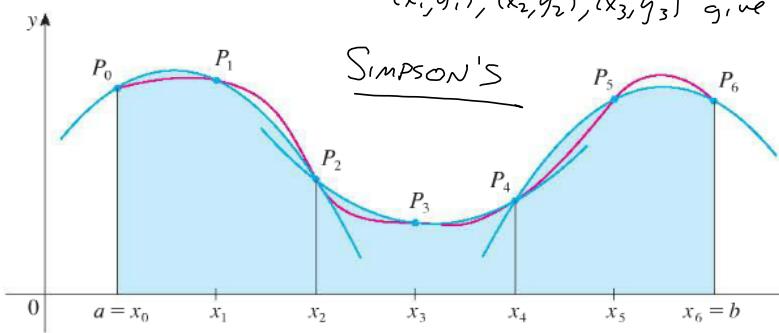


FIGURE 7

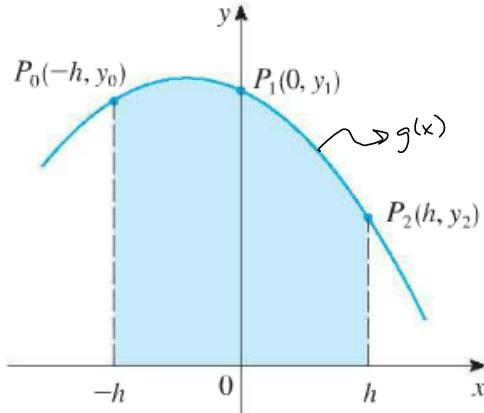
3 variables

$$Ax_1^2 + Bx_1 + C = y_1$$

$$Ax_2^2 + Bx_2 + C = y_2$$

$$Ax_3^2 + Bx_3 + C = y_3$$

which can be solved
for $A, B, \text{ and } C$, if the
3 points aren't on the
same line (non-collinear).



we find the area under one of the parabolas. "n" must be an even number for this to fly.

We show the formula & how it arises in this semi-special case.

$g(x) = Ax^2 + Bx + C$ approximates $f(x)$. We find exact area under $g(x)$. Note that $g(-h) = f(-h)$, $g(0) = f(0)$, $g(h) = f(h)$, because those are the 3 points used to build $g(x)$!

FIGURE 8

Area under $g(x)$:

$$\int_{-h}^h (Ax^2 + Bx + C) dx = \left[\frac{1}{3}Ax^3 + Cx \right]_{-h}^h$$

$$= \frac{1}{3}Ah^3 + Ch - \left(\frac{1}{3}A(-h)^3 + C(-h) \right) = \frac{1}{3}Ah^3 + Ch + \frac{1}{3}Ah^3 + Ch$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

$$= \boxed{\frac{h}{3} [2Ah^2 + 6C]}$$

Observe $(-h, y_0)$ on $f \Rightarrow$

$$A(-h)^2 + B(-h) + C = y_0 \quad \boxed{y_0 = Ah^2 - Bh + C}$$

$(0, y_1)$ on $f \Rightarrow$

$$A(0)^2 + B(0) + C = y_1 \quad \boxed{y_1 = C}$$

(h, y_2) on $f \Rightarrow$

$$Ah^2 + Bh + C = y_2$$

NOTE:

$$y_0 + 4y_1 + y_2 =$$

$$= Ah^2 - Bh + C + 4C + Ah^2 + Bh + C$$

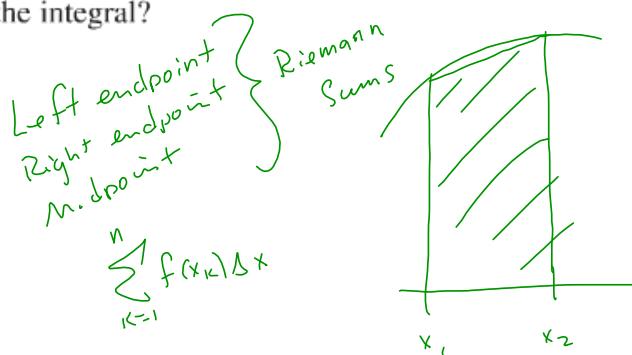
$$= 2Ah^2 + 6C$$

$$\text{So } \frac{h}{3} [2Ah^2 + 6C] = \frac{h}{3} [y_0 + 4y_1 + y_2]$$

Same thing done for x_2, x_3, x_4 gives

$$\frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + y_n]$$

3. Estimate $\int_0^1 \cos(x^2) dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?



$$\begin{aligned} \text{Area} &= \frac{1}{2} (b_1 + b_2) h \\ &= \frac{1}{2} (f(x_1) + f(x_2)) \Delta x \\ &= \frac{f(x_1) \Delta x + f(x_2) \Delta x}{2} \end{aligned}$$