7.3 Trigonometric Substitution

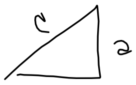
Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

I think doing this cookbook makes it hard. I'd rather just see you draw a triangle, suggested by the radical expression, and use *it* to figure out what to substitute for *x*. If you work it and don't get *x* = sine, secant or tangent expression, switch the lengths on the sides of the leg.

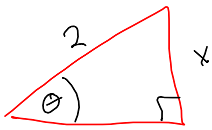
I always go with *x* as the opposite side, unless it's that 3rd case, where the x^2 - a^2 inside the radical makes *x* the length of the hypotenuse.

How do you keep the domains straight? Know the graphs of sine, cosine and tangent. Know how to chop 'em off (restrict their domains) so they're 1-to-1, and so inverses make sense. The rest is knowing your Pythagorean identities from trig.

$\sqrt{4-x^2}$ $x =$ $\sin\theta$, $\tan\theta$, $\sec\theta$ -type stuff.



$a^2+b^2=c^2$
$c^2-a^2=b^2$
$c^2-b^2=a^2$

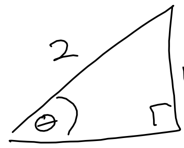


$\sqrt{2^2-x^2}$

$\frac{x}{2} = \sin\theta$

$x = 2\sin\theta$

alternate

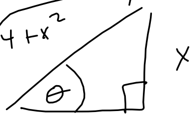


$\frac{x}{2} = \cos\theta$ Meh. $\cos\theta$, $\cos\theta$ sucks.

$x = 2\cos\theta$

$\sqrt{4+x^2}$

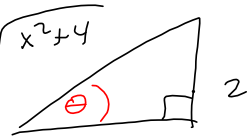
Yes



$\frac{x}{2} = \tan\theta$

$x = 2\tan\theta$

$\sqrt{x^2+4}$

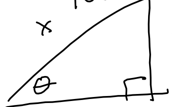


$\frac{x}{2} = \cot\theta$ Nah $\cot\theta$ sucks

$x = 2\cot\theta$

$\sqrt{x^2-5}$

Not as good



$\sqrt{x^2-5}$

Meh.

$\frac{x}{\sqrt{5}} = \csc\theta$

$x = \sqrt{5}\csc\theta$

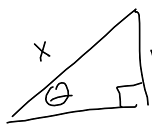
$\frac{\sqrt{5}}{x} = \sin\theta$

$\sqrt{5} = x\sin\theta$

$\frac{\sqrt{5}}{\sin\theta} = x$

$\sqrt{5}\csc\theta = x$

Yes



$\frac{x}{\sqrt{5}} = \sec\theta$

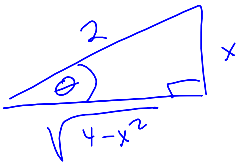
$x = \sqrt{5}\sec\theta$

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ $x = 2\sin\theta$

θ = Restriction to make 1-to-1

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$\frac{x}{2} = \sin\theta$

$x = 2\sin\theta$

$x = 2\sin\theta$

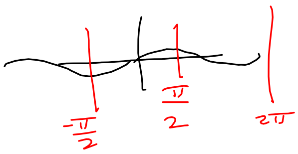
$dx = 2\cos\theta\, d\theta$

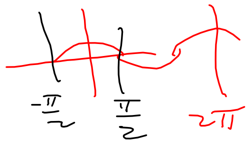
$x^2 = 4\sin^2\theta$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta}$

$= \sqrt{4(1-\sin^2\theta)}$

$= 2\sqrt{\cos^2\theta} = 2|\cos\theta| = 2\cos\theta$





$$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\, 2\cos\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = \frac{1}{2}(-\cot\theta) + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

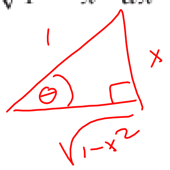
$$\frac{1}{4}\,\frac{(x-2)(x+2)}{x\sqrt{4-x^2}} = \frac{1}{4}\,\frac{x^2-4}{x\sqrt{4-x^2}} = -\frac{1}{4}\,\frac{4-x^2}{x\sqrt{4-x^2}}$$

$$= -\frac{1}{4}\,\frac{4-x^2}{x\sqrt{4-x^2}}\cdot\frac{\sqrt{4-x^2}}{\sqrt{4-x^2}} = -\frac{1}{4}\,\frac{\cancel{(4-x^2)}\sqrt{4-x^2}}{x\cancel{(4-x^2)}}$$

$$= -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x}$$

4–30 Evaluate the integral.

4. $\int_0^1 x^3\sqrt{1-x^2}\,dx$



$x = \sin\theta$

$dx = \cos\theta\,d\theta \longleftrightarrow \dfrac{dx}{d\theta} = \cos\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$

$= \cos\theta$, b/c $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int \frac{\sin^3\theta\cos\theta\,d\theta}{\cos\theta}$$

$$= \int \sin^3\theta\,d\theta$$

$$= \int (1-\cos^2\theta)\sin\theta\,d\theta$$

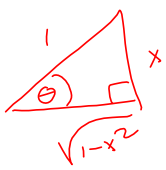
$$= \int \sin\theta\,d\theta + \int \underbrace{\cos^2\theta}_{u^2}\underbrace{(-\sin\theta\,d\theta)}_{du}$$

$$= -\cos\theta + \frac{\cos^3\theta}{3} + C$$

$$\cancel{\sqrt{1-x}}$$

$$\left(\left(1-x^2\right)^{\frac{1}{2}}\right)^3$$

$$= -\sqrt{1-x^2} + \frac{\left(\sqrt{1-x^2}\right)^3}{3} + C = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

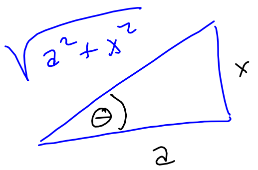


7. $\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}, \quad a > 0$

$(\sqrt{a^2+x^2})^3 = (a^2+x^2)^{\frac{3}{2}}$

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$



$\frac{x}{a} = \tan\theta$

$x = a\tan\theta$

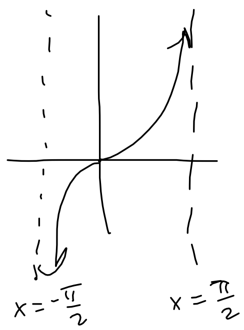
$$\int_0^a \frac{a\sec^2\theta\, d\theta}{\left(\sqrt{a^2+(a\tan\theta)^2}\right)^3}$$

$a^2(1+\tan^2\theta)$

$= a^2\sec^2\theta$

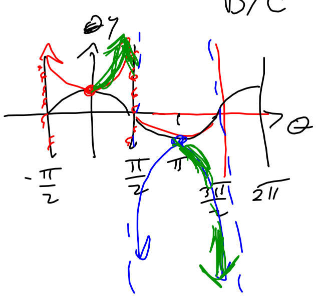
$$= \int_0^a \frac{a\sec^2\theta\, d\theta}{\left(\sqrt{a^2\sec^2\theta}\right)^3} = \int_0^a \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$

$= \frac{1}{a^2}\int_0^a \cos\theta\, d\theta$, etc.



$\sqrt{a^2\sec^2\theta} = a|\sec\theta| = a\sec\theta$

B/C $-\frac{\pi}{2} < x < \frac{\pi}{2}$



$0 \le \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$

11. $\int \sqrt{1-4x^2}\,dx \quad = \frac{1}{2}\int \sqrt{1-u^2}\,du$ & Proceed.

$4x^2 = 2^2x^2 = (2x)^2$

Let $u = 2x$

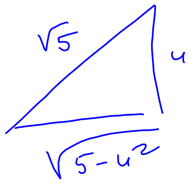
$du = 2dx$

$\frac{du}{2} = dx$

$\sqrt{a^2-x^2}$

12. $\displaystyle\int \frac{du}{u\sqrt{5-u^2}}$

$5=(\sqrt{5})^2$

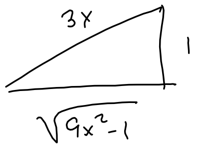


$\frac{u}{\sqrt{5}} = \sin\theta$

$u = \sqrt{5}\sin\theta$, etc.

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} = -\frac{81}{4}+\frac{81}{32}\pi+\frac{567}{64}\sqrt{3}$

$\sqrt{(3x)^2-1^2}$



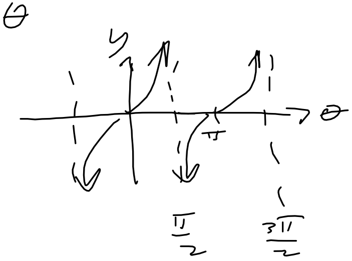
$\frac{3x}{1}=\sec\theta$

$x=\frac{1}{3}\sec\theta \qquad 0\le\theta<\frac{\pi}{2} \text{ or } \pi\le\theta<\frac{3\pi}{2}$

$dx=\frac{1}{3}\sec\theta\tan\theta\,d\theta$

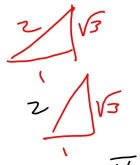
$\sqrt{9\left(\frac{1}{3}\right)^2\sec^2\theta-1}$

$=\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=|\tan\theta|=\tan\theta$



Limits:

$x=\frac{2}{3}=\frac{1}{3}\sec\theta \Rightarrow 2=\sec\theta$



$\theta=\frac{\pi}{3}$

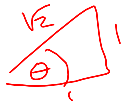
$\displaystyle\int\frac{\frac{1}{3}\sec\theta\tan\theta\,d\theta}{\frac{1}{3^5}\sec^5\theta\ \tan\theta}$

$\displaystyle=3^4\int\sec^{-4}\theta\,d\theta$

$\displaystyle=3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta$

$x=\frac{\sqrt{2}}{3}=\frac{1}{3}\sec\theta$

$\sqrt{2}=\sec\theta$



$\displaystyle=3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta$

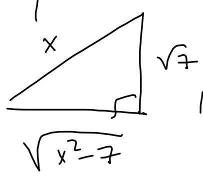
wow!

17. $\int \frac{x}{\sqrt{x^2-7}}\,dx = \int \left(\frac{\sqrt{7}\sec\theta}{\sqrt{7}\tan\theta}\sqrt{7}\sec\theta\tan\theta\right)d\theta = \sqrt{7}\int \sec^2\theta\, d\theta$

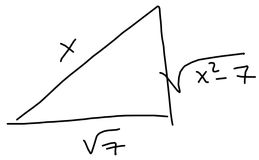
$= \sqrt{7}\tan\theta + C$

$= \sqrt{7}\,\frac{\sqrt{x^2-7}}{\sqrt{7}} + C$

$= \sqrt{x^2-7} + C$



Nah
csc θ sucks



Yeah
sec θ is cool

$\frac{x}{\sqrt{7}} = \sec\theta$

$x = \sqrt{7}\sec\theta$

$dx = \sqrt{7}\sec\theta\tan\theta$

$\sqrt{x^2-7}$

$= \sqrt{7\sec^2\theta - 7}$

$= \sqrt{7}\sqrt{\sec^2\theta - 1}$

$= \sqrt{7}\sqrt{\tan^2\theta}$

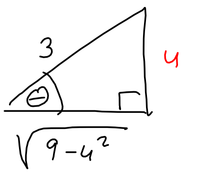
$= \sqrt{7}\,|\tan\theta|$

$= \sqrt{7}\tan\theta$

23. $\int \sqrt{5 + 4x - x^2}\, dx$

$= \int \sqrt{9 - (x-2)^2}\, dx$

$= \int \sqrt{9 - u^2}\, du$, where $u = x - 2$ & $du = dx$



$\frac{u}{3} = \sin\theta$

$u = 3\sin\theta$

$du = 3\cos\theta\, d\theta$

$u = 3\sin\theta$

$\frac{u}{3} = \sin\theta$

$\sin^{-1}\left(\frac{u}{3}\right) = \theta$

$\sin^{-1}\left(\frac{x-2}{3}\right) = \theta$

$-x^2 + 4x + 5$

$\underbrace{-(x^2 - 4x + 2^2}_{-4}) + 5 + 4$

$= -(x-2)^2 + 9$

$\sqrt{9 - (3\sin\theta)^2}$

$= \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)}$

$= \sqrt{9\cos^2\theta} = 3|\cos\theta|$

This gives $\int \sqrt{9 - (3\sin\theta)^2} \cdot 3\cos\theta\, d\theta$

$= 3\int 3|\cos\theta|\cos\theta\, d\theta$

$\cos\theta \geq 0$ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$= 9\int \cos^2\theta\, d\theta$

$= \frac{9}{2}\int (1 + \cos(2\theta))\, d\theta$

$= \frac{9}{2}\int d\theta + \frac{9}{2}\cdot\frac{1}{2}\int \cos(2\theta)\, 2d\theta$

$= \frac{9}{2}\theta + \frac{9}{4}(\sin(2\theta)) + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{4}(2\sin\theta\cos\theta) + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{4}\cdot 2\left(\frac{u}{3}\cdot\frac{\sqrt{9-u^2}}{3}\right) + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}\left((x-2)\sqrt{9 - (x-2)^2}\right) + C$

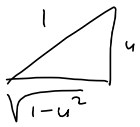
$-\frac{1}{4}(-2x + 4)\sqrt{5 - x^2 + 4x} + \frac{9}{2}\arcsin\left(\frac{1}{3}x - \frac{2}{3}\right)$

29. $\int x\sqrt{1-x^4}\,dx = \int x\sqrt{1-(x^2)^2}\,dx$

$u = x^2$, then $du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$= \frac{1}{2}\int\sqrt{1-u^2}\;2x\,dx$

$= \frac{1}{2}\int\sqrt{1-u^2}\,du = \frac{1}{2}\int\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \frac{1}{2}\int\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$



$= \frac{1}{2}\int|\cos\theta|\cos\theta\,d\theta = \frac{1}{2}\int\cos^2\theta\,d\theta$

$= \frac{1}{2}\cdot\frac{1}{2}\left(\int\left(1-\cos(2\theta)\right)d\theta\right)$

$u = \sin\theta \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = \cos\theta\,d\theta$

etc.

$u = \boxed{\tan^{-1}(\theta)}$

$\sin^{-1}(u) = \theta$

$= \frac{1}{4}\int d\theta - \frac{1}{4}\cdot\frac{1}{2}\int\cos(2\theta)\,2\,d\theta$

$\tan\theta$

$\sin^{-1}(x^2) = \theta$

$= \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta) + C$

$\tan^{-1}(\tan\theta)$

$= \theta$

$= \frac{1}{4}\theta - \frac{1}{8}\cdot 2\sin\theta\cos\theta + C$

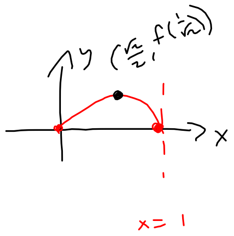
$\sin(\theta+\omega)$

$= \frac{1}{4}\sin^{-1}(x^2) - \frac{1}{4}x^2\sqrt{1-(x^2)^2}$

$= \sin\theta\cos\omega + \sin\omega\cos\theta$

38. Find the volume of the solid obtained by rotating about the line $x = 1$ the region under the curve $y = x\sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.

↳ Top $\frac{1}{2}$ of circle, multiplied by x



$$y' = 1(1-x^2)^{\frac{1}{2}} + x\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

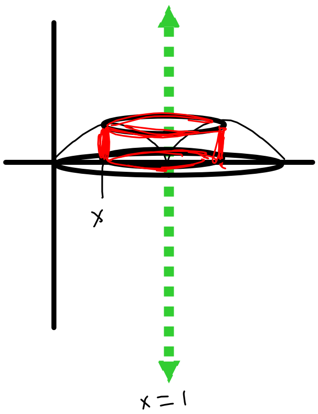
$$= \frac{1-2x^2}{\sqrt{1-x^2}}$$

undefined at ± 1

slope $= 0$ when $-2x^2+1=0$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

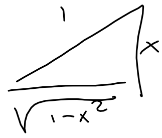


$$2\pi\int_0^1 r\,h\,\Delta x$$

$$= 2\pi\int_0^1 \underbrace{(1-x)}_{r}\,\underbrace{x\sqrt{1-x^2}}_{h}\,\underbrace{dx}_{\Delta x}$$

$$= 2\pi\int_0^1 x\sqrt{1-x^2}\,dx - 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx$$

$$= -2\pi\cdot\frac{1}{2}\int_0^1 \sqrt{1-x^2}\cdot(-2x)\,dx$$



$u = 1-x^2$

$$= -\pi\left[\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - 2\pi\int_{x=0}^{x=1} \sin^2\theta\,\cos^2\theta\,d\theta$$

$x = \sin\theta$

$$= -\frac{2\pi}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1 - \frac{\pi}{4}\int_{x=0}^{x=1}(1-\cos(4\theta))\,d\theta$$

$$\frac{1}{4}(1-\cos^2(2\theta))$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2}(1+\cos(4\theta))\right)$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$= \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$= -\frac{2\pi}{3}\left[0^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] - \frac{\pi}{4}\int_{x=0}^{x=1} d\theta + \frac{\pi}{4}\cdot\frac{1}{4}\int_{x=0}^{x=1}\cos(4\theta)4\,d\theta$$

$$= \frac{2\pi}{3} - \frac{\pi}{4}\theta + \frac{\pi}{8}\sin\theta\Big]_{x=0}^{x=1} = \frac{2\pi}{3} - \frac{\pi}{4}\arcsin(x)\Big]_0^1 + \frac{\pi}{8}x\Big]_0^1$$

$= 0$

$$= \frac{2\pi}{3} - \frac{\pi}{4}\left[\arcsin(1) - \arcsin(0)\right] + \frac{\pi}{8}[1-0]$$

$$= \frac{2\pi}{3} - \frac{\pi}{4}\cdot\frac{\pi}{2} + \frac{\pi}{8}$$

$=$