Section 7.2 Trigonometric Integrals

This is just like #47, Section 7.1. We now work it the way I alluded to, but never got back to:

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \int \left[\tfrac{1}{2}(1 - \cos(2x))\right]^2 dx$$

$$= \tfrac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right] dx$$

$\tfrac{1}{2}(1 + \cos(4x))$

$$= \tfrac{1}{4}\int dx - \tfrac{1}{2}\cdot\tfrac{1}{2}\int \cos(2x)\,2\,dx + \tfrac{1}{4}\int \tfrac{1}{2}(1 + \cos(4x))\,dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{2}\cdot\tfrac{1}{2}\sin(2x) + \tfrac{1}{8}\int dx + \tfrac{1}{8}\cdot\tfrac{1}{4}\int \cos(4x)\cdot 4\,dx$$

$$= \boxed{\tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{8}x + \tfrac{1}{32}\sin(4x) + C}$$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x\,(\cos^2 x)^k \cos x\,dx$$

$$= \int \sin^m x\,(1 - \sin^2 x)^k \cos x\,dx$$

$$= \int \sin^{*} x \cdot \underbrace{\cos x\,dx}_{du} = \frac{\sin^{*+1}(x)}{*+1}$$

Then substitute $u = \sin x$.

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\,dx = \int (\sin^2 x)^k \cos^n x \sin x\,dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\,dx$$

$du = -\sin x\,dx$
$u = \cos x$
} Play with negative signs.

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\frac{1 + \cos(2x)}{2}$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

$\sin(2x) = 2\sin x \cos x$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

$\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\sec x] = \underline{\sec x \tan x}$

1. $\int \sin^2 x \cos^3 x \, dx$

$\int \sin^n(u) \cdot \cos(u) \, du = \int v^n \, dv$

$= \int \sin^2 x \cos^2 x \overbrace{\cos x \, dx}^{du}$

scratch

$\sin^2 x (1 - \sin^2 x) \cos x$

$= \sin^2 x \cos x - \sin^4 x \cos x$

$= \int \sin^2 x \cos x \, dx - \int \sin^4 x \cos x \, dx$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$

4. $\int_0^{\pi/2} \sin^5 x\,dx$. $\int \sin^4 x \cdot \sin x\,dx = -\int \sin^4 x\,(-\sin x\,dx)$

?

$= -\int (1-2\cos^2 x+\cos^4 x)(-\sin x\,dx)$ $\qquad \sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2$

?

$= +\int +\sin x\,dx + \int(+2\cos^2 x)(-\sin x\,dx) - \int \cos^4 x\,(-\sin x\,dx$ $\qquad = 1-2\cos^2 x+\cos^4 x$

oops! Definite Integral

$= -\cos x + 2\cdot\frac{1}{3}\cos^3(x) - \frac{1}{5}\cos^5 x + C$

$\therefore \int_0^{\frac{\pi}{2}} \sin^5 x\,dx = -\cos(x)\Big]_0^{\frac{\pi}{2}} + \frac{2}{3}\cos^3(x)\Big]_0^{\frac{\pi}{2}} - \frac{1}{5}\cos^5(x)\Big]_0^{\frac{\pi}{2}}$

$= -\cos(\frac{\pi}{2}) - (-\cos(0)) + \frac{2}{3}\cos^3(\frac{\pi}{2}) - \frac{2}{3}\cos^3(0) - \frac{1}{5}\cos^5(\frac{\pi}{2}) - (-\frac{1}{5}\cos^5(0))$

$= 0 + 1 + 0 - \frac{2}{3} - 0 + \frac{1}{5}$

$= \frac{15-10+3}{15} = \boxed{\frac{8}{15}}$

6. $\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$

Synthesis of trig tech & u-substitution

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx \Longrightarrow \frac{1}{2}x^{-\frac{1}{2}}\,dx = du$

$dx = \frac{du}{\frac{1}{2}x^{-\frac{1}{2}}} = 2x^{\frac{1}{2}}du$ This gives

$\int \frac{\sin^3(x^{\frac{1}{2}})}{\cancel{x^{\frac{1}{2}}}} \cdot 2\cancel{x^{\frac{1}{2}}}du$

$= 2\int \sin^3(u)\,du$

$2\int \sin^3(x^{\frac{1}{2}}) \cdot \frac{1}{2}x^{-\frac{1}{2}}dx$

$= 2\int \sin^3 u\,du = 2\int \underline{\sin^2 u} \cdot \sin u\,du$

convert to $1-\cos^2 x$

10. $\int_0^{\pi} \sin^2 t \cos^4 t \, dt \Rightarrow$

(Power) Reduction Formula.

$$\sin^2 t = \frac{1}{2}(1-\cos(2t))$$

$$\cos^4 t = (\cos^2 t)^2 = \left(\frac{1}{2}(1+\cos(2t))\right)^2 =$$

$$= \frac{1}{4}(1+2\cos(2t)+\cos^2(2t))$$

$$\frac{1}{8}\int (1-\cos(2t))(1+2\cos(2t)+\cos^2(2t))\,dt$$

$$= \frac{1}{8}\int\left(1+2\cos(2t)+\cos^2(2t) - \cos(2t) - 2\cos^2(2t) - \cos^3(2t)\right)dt$$

$$= \frac{1}{8}\int\left(1+\cos(2t) - \cos^2(2t) - \cos^3(2t)\right)dt$$

$$= \frac{1}{8}\int dt + \frac{1}{8}\cdot\frac{1}{2}\int \cos(2t)\,2\,dt - \frac{1}{8}\cdot\frac{1}{2}\cdot\frac{1}{4}\int(1+\cos(4t))\,4\,dt$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2t) - \frac{1}{64}\int 4\,dt - \frac{1}{64}\sin(4t) + C$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2t) - \frac{1}{16}t - \frac{1}{64}\sin(4t) + C \Rightarrow$$

$\int_0^{\pi}(*)$ is ready, now.

14. $\int \cos\theta \cos^5(\sin\theta)\, d\theta$

$$= \int \cos^5(\sin\theta) \cdot \cos\theta \, d\theta$$

$$= \int \cos^5(u)\, du$$

$$= \int \cos^4(u) \cdot \cos(u)\, du$$

$$= \int (\cos^2(u))^2 \cos(u)\, du$$

$$= \int (1-\sin^2(u))^2 \cos(u)\, du$$

$$= \int (1-2\sin^2(u) + \sin^4(u)) \cos(u)\, du$$

$$= \int \cos(u)\, du - 2\int \sin^2(u)\cos(u)\, du + \int \sin^4(u)\cos(u)\, du$$

$$= \sin(u) - 2\int v^2\, dv + \int v^4\, dv$$

$$= \sin(u) - \frac{2}{3}v^3 + \frac{1}{5}v^5 + C$$

$$= \sin(\sin\theta) - \frac{2}{3}\sin^3(\sin\theta) + \frac{1}{5}\sin^5(\sin\theta) + C$$

17. $\int \cos^2 x \tan^3 x \, dx$

$$\cos^2 x \frac{\sin^3 x}{\cos^3 x} = \frac{\sin^3 x}{\cos x}$$

$$= \frac{\sin^2 x}{\cos x} \cdot \sin x$$

$$= \frac{1-\cos^2 x}{\cos x} \cdot \sin x = \left(\frac{1}{\cos x} - \cos x\right) \sin x$$

$$= \int \left(\tan x - \cos x \underbrace{\sin x}_{u}\right) \underbrace{dx}_{du}$$

$$\int u \, du = \frac{1}{2} u^2 + C$$

$$= \ln|\sec x| - \frac{\sin^2 x}{2} + C$$

25. $\int \tan^4 x \sec^6 x \, dx$

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$

$\sec^2 x = \tan^2 x + 1$

$= \int \tan^4 x \sec^4 x \cdot \sec^2 dx$

$= \int \tan^4 x (\tan^2 x + 1)^2 \cdot \sec^2 x \, dx$

$= \int \tan^4 x (\tan^4 x + 2\tan^2 x + 1) \sec^2 x \, dx$

$= \int (\tan^8 x + 2\tan^6 x + \tan^4 x) \sec^2 x \, dx$

$= \int (u^8 + 2u^6 + u^4) \, du$, where $u = \tan x$

$= \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C$

26. $\int_0^{\pi/4} \sec^4\theta \ \tan^4\theta \, d\theta$

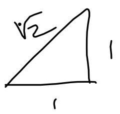
$$= \int_0^{\frac{\pi}{4}} \sec^2\theta \tan^4\theta \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + 1) \tan^4\theta \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^6\theta \sec^2\theta \, d\theta + \int_0^{\frac{\pi}{4}} \tan^4 \sec^2\theta \, d\theta$$

$$= \frac{1}{7}\tan^7\theta \Big]_0^{\frac{\pi}{4}} + \frac{1}{5}\tan^5\theta \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{7}(1)^7 - 0 + \frac{1}{5}(1)^5 - 0 = \frac{1}{7} + \frac{1}{5} = \frac{5+7}{35} = \boxed{\frac{12}{35}}$$



Sribbles.

$\sec^2 = \tan^2 + 1$

$\csc^2 = \frac{1}{\sin^2} \cdot \frac{\cos^2}{\cos^2} = \cot^2 \sec^2 = \cot^2 + \cot^2 \tan^2$

$= \cot^2 + 1$

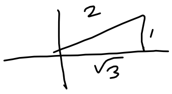
$\therefore \csc^2 - 1 = \cot^2$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc^2 x - 1)\, dx$

$= \Big[-\cot x - x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = -\cot\left(\frac{\pi}{2}\right) - \frac{\pi}{2} - \left(-\cot\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right)$

$= -0 - \frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6} = \boxed{\sqrt{3} - \frac{\pi}{3}}$



36. $\int_{\pi/4}^{\pi/2} \cot^3 x\, dx$ See $\int \tan^3 x\, dx$, Example 7 pg 499

$\int \cot^2 x \cot x\, dx = \int (\csc^2 x - 1)\cot x\, dx$

$= \int \csc^2 x \cot x\, dx - \int \cot x\, dx$

$= -\int (\csc x)(-\csc x \cot x\, dx) - \ln|\sin x|$

$\boxed{= -\frac{1}{2}\csc^2 x - \ln|\sin x| + C}$

$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

$= \ln|\sin x| + C$

(48) $$\int \frac{dx}{\cos x - 1} = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx$$

$$= \int \frac{\cos x + 1}{-\sin^2 x}\,dx = -\int\left(\frac{\cos x}{\sin^2 x} + \frac{1}{\sin^2 x}\right)dx$$

$$= -\int \sin^{-2} x \cdot \cos x\,dx - \int \csc^2 x\,dx$$

$$= -\frac{\sin^{-1}}{-1} + \cot^2 x + C$$