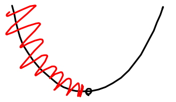
1. (10 pts) The function $f(x) = x^2 - 6x - 11$ is 1-to-1 on the restricted domain $[3, \infty)$. Find the inverse function. State its domain and range.

$\text{vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = (3, f(3)) = (3, -20)$



$(3, -20)$

$\mathcal{D}(f) = [3, \infty)$
$= \mathcal{R}(f^{-1})$

$$\begin{array}{r|rrr} 3 & 1 & -6 & -11 \\ & & 3 & -9 \\ \hline & 1 & -3 & -20 \end{array}$$

$\mathcal{R}(f) = [-20, \infty)$
$= \mathcal{D}(f^{-1})$

$y^2 - 6y - 11 = x$

$y^2 - 6y = x + 11$

$y^2 - 6y + 3^2 = x + 11 + 9$

$(y-3)^2 = x + 20$

$y - 3 = \pm\sqrt{x+20} \Longrightarrow y = \boxed{f^{-1}(x) = 3 + \sqrt{x+20}}$

2. Find $(f^{-1})'(5)$ for $f(x) = x^2 - 6x - 11 \quad (x \geq 3)$, in two ways:
   a. (5 pts) Directly, using your answer from #1.
   b. (5 pts) Using our theorem for derivative of the inverse..

(a) $f^{-1}(x) = 3 + \sqrt{x+20} = (x+20)^{\frac{1}{2}} + 3 \Longrightarrow (f^{-1})'(x) = \frac{1}{2}(x+20)^{-\frac{1}{2}}$

$\Longrightarrow (f^{-1})'(5) = \frac{1}{2}(5+20)^{-\frac{1}{2}} = \frac{1}{2}(25)^{-\frac{1}{2}} = \frac{1}{2(25)^{\frac{1}{2}}} = \frac{1}{2(5)} = \boxed{\frac{1}{10} = (f^{-1})'(5)}$

(b) $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))}$

$f(x) = 5$
$x^2 - 6x - 11 = 5$
$x^2 - 6x - 16 = 0$
$(x-8)(x+2) = 0$
$x = 8$ OR $x = -2 \notin \mathcal{D}$
$\Rightarrow \in$ restricted $\mathcal{D}$

$f(x) = x^2 - 6x - 11$
$\Longrightarrow f'(x) = 2x - 6$
$\Longrightarrow f'(f^{-1}(5)) = f'(8) = 2(8) - 6$
$= 16 - 6$
$= 10$

$\Longrightarrow \boxed{(f^{-1})'(5) = \frac{1}{10}}$

3. (5 pts each) Find the derivative with respect to x. Do not simplify.

$b^x = e^{\ln(b^x)}$

$\frac{d}{dx}\left[b^x\right] = (\ln(b))\, b^x$

a. $y = 2 \cdot 3^{2x^2-3x}$

$= \left(2 \cdot \ln(3) \cdot 3^{2x^2-3x}\right)(4x-3)$

$y = 2 \cdot e^{\ln(3^{2x^2-3x})} = 2 \cdot e^{(2x^2-3x)\ln(3)} \Rightarrow y' = 2 \cdot e^{\ln(3)(2x^2-3x)} \cdot \ln(3)(4x-3)$

$= 2e^{\ln(3^{2x^2-3x})} \cdot \ln(3)(4x-3)$

$= \left(2 \cdot 3^{2x^2-3x}\right)(\ln(3)(4x-3))$

b. $y = \ln\left(\frac{(x^2-2x)^3}{(2x+1)^5}\right)$ (Hint: Break it up into simpler logs!)

$= \ln\left((x^2-2x)^3\right) - \ln\left((2x+1)^5\right)$

$= 3\ln(x^2-2x) - 5\ln(2x+1) \Rightarrow$

$$\boxed{y' = 3\left(\frac{2x-2}{x^2-2x}\right) - 5\left(\frac{2}{2x+1}\right)}$$

c. $y = \log_3(x^2-2x)$

Definitely easier to just use change of base, rather than memorize

$\frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{\ln(b)} \cdot \frac{1}{x}$

$= \frac{1}{\ln(3)} \ln(x^2-2x)$

$\Rightarrow \boxed{y' = \frac{1}{\ln(3)}\left(\frac{2x-2}{x^2-2x}\right)}$ was quicker.

$y = \frac{1}{\ln(3)}\left[\ln(x) + \ln(x-2)\right]$

$\Rightarrow y' = \boxed{\frac{1}{\ln(3)}\left[\frac{1}{x} + \frac{1}{x-2}\right]} = \frac{1}{\ln(3)}\left[\frac{x-2+x}{x(x-2)}\right]$

$= \frac{1}{\ln(3)}\left[\frac{2x-2}{x(x-2)}\right]$

d. $y = (x^2-3x)^{2x^2+5x}$ Logarithmic Diff. $f(x)^{g(x)}$ s.t.c.

$\ln(y) = \ln\left((x^2-3x)^{2x^2+3x}\right) = (2x^2+3x)\ln(x^2-3x)$

$\Rightarrow \frac{y'}{y} = (4x+3)\ln(x^2-3x) + (2x^2+3x)\left(\frac{2x-3}{x^2-3x}\right)$

$(fg)' = \underbrace{f'g + g'f}_{\text{OK}} = \underline{f'g + fg'}$ nicer

e. $y = x^2 \sin^{-1}(x^2 - 3x)$

$\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \boxed{y' = 2x \sin^{-1}(x^2-3x) + x^2\left(\frac{1}{\sqrt{1-x^2}}\right)}$

f. $y = x^2 \tanh^{-1}(x^2 - 3x)$

hyperbolics de-emphasized.

$\Rightarrow \boxed{y' = 2x \tanh^{-1}(x^2-3x) + x^2\left(\frac{1}{1-(x^2-3x)^2}\right)(2x-3)}$

Chain Rule!

$\frac{d}{dx}\left[\tanh^{-1}(x)\right] = \frac{1}{1-x^2}$

$\frac{d}{dx}\left[\tanh^{-1}(u)\right] = \frac{1}{1-u^2} \cdot \frac{du}{dx}$

4. (5 pts each) Evaluate the integral.

a. $\int (2x-3)e^{x^2-3x}dx$

$\int e^u du = e^u + C$

$u = x^2 - 3x$

$\boxed{du = (2x-3)\,dx} \Rightarrow \frac{du}{2x-3} = dx$

$= \int \cancel{(2x-3)} e^u \frac{du}{\cancel{2x-3}} = \int e^u du = e^u + C = e^{x^2-3x} + C$

$\int e^u (2x-3)dx = \int e^u du$, etc.

b. $\int \frac{dx}{x\sqrt{9-x^2}}$

arcsech(x)

$\int \frac{dx}{x\sqrt{1-x^2}} = \operatorname{sech}^{-1}(x) + C$

$= \int \frac{dx}{x\sqrt{9(1-\frac{x^2}{9})}} = \int \frac{dx}{3x\sqrt{1-(\frac{x}{3})^2}} = \int \frac{dx}{\sqrt{1-u^2}} = \ldots = \operatorname{sech}^{-1}(u) + C$

$\frac{x^2}{9} = \left(\frac{x}{3}\right)^2$

$= \int \frac{\cancel{3}du}{\cancel{3}(3(\frac{x}{3}))\sqrt{1-u^2}}$

$u = \frac{x}{3}$

$du = \frac{1}{3}dx$

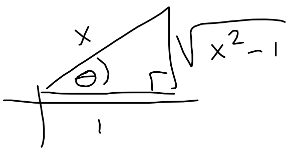
$3du = dx$

$= \frac{1}{3}\int \frac{du}{u\sqrt{1-u^2}} = \frac{1}{3}\operatorname{sech}^{-1}(u) + C$

$\boxed{= \frac{1}{3}\operatorname{sech}^{-1}\left(\frac{x}{3}\right) + C}$

5. (5 pts each) Simplify:

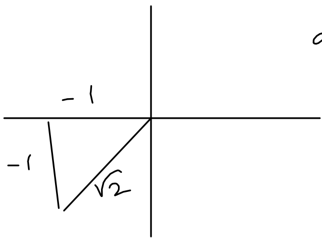
a. $\tan(\sec^{-1}(x)) = \tan\theta = \sqrt{x^2-1}$



b. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

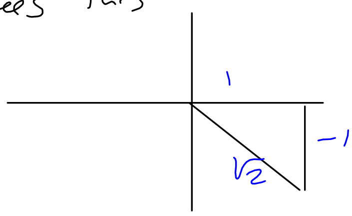
$= \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\boxed{= -\frac{\pi}{4}}$



arcsine $= \sin^{-1}$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

so $\sin^{-1}(-\frac{1}{\sqrt{2}})$ only sees this



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uninhibited growth/decay.

6. The half-life of Carbon-14 is about 5730 years. How old is a fire pit in which 30% of the original Carbon-14 remains?

Pre-Calc. $A(t) = A_0 e^{kt}$

$A_0 = A(0)$, since

$A(0) = A_0 e^{k\cdot 0} = A_0 e^0 = A_0$

$A(5730) = A_0 e^{k\cdot 5730} = \frac{1}{2} A_0$

$e^{5730k} = \frac{1}{2}$

$\ln(\quad) = \ln(\quad)$

$5730k = \ln(\frac{1}{2}) = -\ln(2)$

$k = -\frac{\ln(2)}{5730}$

$A(t) = .3A_0$

$A_0 e^{kt} = .3A_0$

$e^{kt} = .3$

$kt = \ln(.3) \Longrightarrow t = \frac{\ln(.3)}{k} = \frac{\ln(.3)}{-\frac{\ln(2)}{5730}}$

$\boxed{t = -\frac{\ln(.3)(5730)}{\ln(2)}} \approx$

7. Evaluate $\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 3x}$ in two ways:

a. Factor, cancel, pass to the limit.

b. L'Hopital's rule

(a) $\frac{x^2-x-6}{x^2-3x} = \frac{(\cancel{x-3})(x+2)}{x(\cancel{x-3})} = \frac{x+2}{x} = \frac{\cancel{x}(1+\frac{2}{x})}{\cancel{x}(1)} \xrightarrow{x\to\infty} \frac{1}{1} \boxed{= 1}$ $\frac{2}{x} \to 0$

(b) $\frac{x^2-x-6}{x^2-3x} \xrightarrow{x\to\infty} \frac{\infty}{\infty}$ so L'Hôpital's = L'H

$\frac{x^2-x-6}{x^2-3x} \xrightarrow[x\to\infty]{L'H} \frac{2x-1}{2x-3} \xrightarrow[x\to\infty]{L'H} \frac{2}{2} \boxed{= 1}$

$\sinh(0) = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$

8. Evaluate the limits:

$\cosh(0) = \dfrac{e^0 + e^0}{2} = \dfrac{2}{2} = 1$

a. $\lim\limits_{x\to 0}\left(\dfrac{\sinh(x) - x}{x^3}\right) \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{\cosh(x) - 1}{3x^2} \overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{\sinh(x)}{6x} \overset{l'H}{=}$

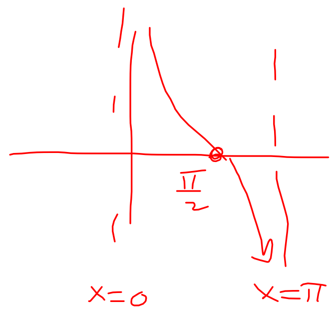
$= \lim\limits_{x\to 0} \dfrac{\cosh(x)}{6} = \boxed{\dfrac{1}{6}}$

b. $\lim\limits_{x\to\infty}\left(x \sin\left(\dfrac{3}{x}\right)\right) = \infty \cdot 0$ $\qquad \dfrac{3}{x} \xrightarrow{x\to\infty} 0$

$\dfrac{d}{dx}\left[x^{-1}\right] = -1\, x^{-2}$

$= \lim\limits_{x\to\infty} \dfrac{\sin\left(\frac{3}{x}\right)}{\frac{1}{x}} \overset{l'H}{=} \lim\limits_{x\to\infty} \dfrac{\cos\left(\frac{3}{x}\right)\left(+\frac{3}{\cancel{x^2}}\right)}{+\frac{1}{\cancel{x^2}}}$

$= \dfrac{(1)(3)}{1} \boxed{= 3}!$



c. $\lim\limits_{x\to\infty}(4x+1)^{\cot(x)} = 1^{\infty}$

$\to x \to 0^+$

$y = (4x+1)^{\cot(x)}$

$\ln(y) = (\cot(x))\ln(4x+1)$

$\lim\limits_{x\to 0^+}(\ln(y)) = \lim\limits_{x\to 0^+} \cot(x)\ln(4x+1) = \infty \cdot 0 = \dfrac{0}{\frac{1}{\infty}} = \dfrac{0}{0}$

$= \lim\limits_{x\to 0^+} \dfrac{\ln(4x+1)}{\tan(x)} \overset{l'H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{4}{4x+1}}{\sec^2(x)} = \dfrac{\frac{4}{1}}{1}$ $\quad = 4 = \lim\limits_{x\to 0^+}(\ln(y))$

$\Rightarrow \lim\limits_{x\to 0^+} y \boxed{= e^4} \to$ Don't forget we took $\ln$ (both sides).