

$\leq 6.8 \#s$  1-4, 5, 7, 10, 13, 16, 19, 23, 26, 30,  
41, 43\*, 54, 58 ~~\*~~

Sep 15-1:48 PM

## 6.8 Indeterminate Forms and L'Hospital's Rule

If numerator & denominator approach the same value, then we see how FAST each is doing so!

If  $\frac{a}{b} \xrightarrow{x \rightarrow c} \frac{0}{0}$  or  $\frac{\infty}{\infty}$  or  $\frac{\infty^0}{0^\infty}$  (All indeterminate forms)  
 then  $\frac{da}{dx}$  can help (in 1st 2 cases).  $\frac{db}{dx}$    
 ln (Both Sides)

$$\boxed{E} \quad \lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \frac{0}{0}$$

$$\text{L'Hopital says: } \frac{d}{dx} [\ln x] = \frac{1}{x}$$

$$\frac{d}{dx} [x-1] = 1$$

$$\text{and } \lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = \boxed{1}$$

Sep 15-11:46 AM

**L'Hospital's Rule** Suppose  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval  $I$  that contains  $a$  (except possibly at  $a$ ). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

$$\text{or that} \quad \lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type  $\frac{0}{0}$  or  $\infty/\infty$ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is  $\infty$  or  $-\infty$ ).

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

Sep 15 1:27 PM

$0 \cdot \infty$  is not  $0 \cdot \infty$

It's  $\frac{1}{\infty} \cdot \infty$  OR  $0 \cdot \frac{1}{0}$

$$\text{I.e.; } \lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty)$$

$$\text{Iny: } \frac{\ln x}{\frac{1}{x}} \stackrel{\infty}{\infty} \text{ OR } \frac{x}{\frac{1}{\ln x}} \stackrel{0}{0}$$

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$$

$$\begin{aligned} & \text{L'H} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = \boxed{0} \\ & - \frac{1}{x} \cdot \frac{x^2}{1} = -x \end{aligned}$$

Sep 15 1:29 PM

$\infty - \infty = ?$

$\boxed{EB}$

$\lim_{x \rightarrow (\frac{\pi}{2})^-} (\sec x - \tan x) = \infty - \infty$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

 $= \lim_{x \rightarrow \frac{\pi}{2}^-} \left( \frac{1 - \sin x}{\cos x} \right) \text{ is } \frac{0}{0}$ 
 $= \lim_{x \rightarrow \frac{\pi}{2}^-} \left( \frac{-\cos x}{-\sin x} \right) = \boxed{0}$

Sep 15 1:33 PM

$y = 0^\circ$  or  $0^\infty$  or  $\infty^0$  or ... Take  $\ln$  both sides.

$\ln y = \text{etc. Gets } 0 \text{ or } \infty \text{ out of exponent.}$

$\lim_{x \rightarrow 1^+} x^{\frac{1}{1-x}}$        $y = x^{\frac{1}{1-x}}$        $\infty \cdot 0$

$\ln y = \ln(x^{\frac{1}{1-x}}) = \frac{1}{1-x} \ln(x)$

$= \frac{\ln x}{1-x} \stackrel{L'H}{=} \frac{\frac{1}{x}}{-1} \rightarrow -1$

$\lim_{x \rightarrow 1^+} (\ln y) = -1$

$\ln(\lim_{x \rightarrow 1^+} y) = -1$

$\lim_{x \rightarrow 1^+} y = e^{-1} \boxed{= \frac{1}{e}}$

$e \quad e$

Sep 15 1:35 PM

**3 Cauchy's Mean Value Theorem** Suppose that the functions  $f$  and  $g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $g'(x) \neq 0$  for all  $x$  in  $(a, b)$ . Then there is a number  $c$  in  $(a, b)$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Sep 15 1:44 PM

**1-4** Given that

$$\begin{aligned}\lim_{x \rightarrow a} f(x) &= 0 & \lim_{x \rightarrow a} g(x) &= 0 & \lim_{x \rightarrow a} h(x) &= 1 \\ \lim_{x \rightarrow a} p(x) &= \infty & \lim_{x \rightarrow a} q(x) &= \infty\end{aligned}$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a)  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  ✓      (b)  $\lim_{x \rightarrow a} \frac{f(x)}{p(x)}$  No      (c)  $\lim_{x \rightarrow a} \frac{h(x)}{p(x)}$  No  
 (d)  $\lim_{x \rightarrow a} \frac{p(x)}{f(x)}$  No      (e)  $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$  ✓

$$\lim_{x \rightarrow a} f(x) = 0 \quad \lim_{x \rightarrow a} g(x) = 0 \quad \lim_{x \rightarrow a} h(x) = 1$$

$$\lim_{x \rightarrow a} p(x) = \infty \quad \lim_{x \rightarrow a} q(x) = \infty$$

2. (a)  $\lim_{x \rightarrow a} [f(x)p(x)]$  ✓      (b)  $\lim_{x \rightarrow a} [h(x)p(x)]$  =  $\infty$   
 (c)  $\lim_{x \rightarrow a} [p(x)q(x)]$  =  $\infty$

$$4. \lim_{x \rightarrow a} [f(x)]^{g(x)}$$

$$(d) \lim_{x \rightarrow a} [p(x)]^{f(x)}$$

$$(e) \lim_{x \rightarrow a} [p(x)]^{q(x)}$$

$$(f) \lim_{x \rightarrow a} \sqrt[q(x)]{p(x)}$$

1. (b)  $\lim_{x \rightarrow a} \frac{f(x)}{p(x)}$  is of

the form  $\frac{\infty}{\infty}$  & that's  $\boxed{0}$

(c)  $\lim_{x \rightarrow a} \frac{h(x)}{p(x)}$  is  $\frac{1}{\infty} = \boxed{0}$

(d)  $\lim_{x \rightarrow a} \frac{p(x)}{f(x)}$  is  $\frac{\infty}{\infty} = \boxed{\infty}$

3. (a)  $\lim_{x \rightarrow a} [f(x) - p(x)]$  =  $\infty$  ✓      (b)  $\lim_{x \rightarrow a} [p(x) - q(x)]$  ✓

(c)  $\lim_{x \rightarrow a} [p(x) + q(x)]$  No

$$\infty + \infty = \infty$$

$$\lim_{x \rightarrow a} \frac{1}{p(x)^{f(x)}} = \infty$$

Sep 15 1:54 PM

$$7. \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - x} = \frac{0}{0}$$

$$\begin{aligned} \frac{x^2 - 1}{x^2 - x} &= \frac{(x-1)(x+1)}{x(x-1)} \\ &= \frac{x+1}{x} \xrightarrow{x \rightarrow 1} \frac{1+1}{1} = 2 \end{aligned}$$

L'Hopital's Rule

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - x} \stackrel{L'H}{=} \lim_{x \rightarrow 1} \frac{2x}{2x-1} = \frac{2(\frac{1}{2})}{2(\frac{1}{2})-1} = \frac{2}{1} = 2 \quad \begin{aligned} &4x^2 + 8x - 2x - 9 \\ &= 2x(2x+9) - 1(2x+9) \\ &= (2x+9)(2x-1) \end{aligned}$$

See next page!

Turned '16' INTO '18'

magically.

$$\frac{6x^2 + 5x - 4}{4x^2 + 8x - 9} = \frac{(3x+4)(2x-1)}{(2x+9)(2x-1)} = \frac{3x+4}{2x+9} \xrightarrow{x \rightarrow \frac{1}{2}} \frac{17}{20}$$

L'Hopital's way;  
Ascertain its  $\frac{0}{0}$  switch, then  
differentiate.

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}} \frac{6x^2 + 5x - 4}{4x^2 + 8x - 9} &\stackrel{L'H}{=} \lim_{x \rightarrow \frac{1}{2}} \frac{12x+5}{8x+16} = \frac{12(\frac{1}{2})+5}{8(\frac{1}{2})+16} \\ &= \frac{6+5}{4+16} = \frac{11}{20} \end{aligned}$$

Sep 15-2:47 PM

$$10. \lim_{x \rightarrow 1/2} \frac{6x^2 + 5x - 4}{4x^2 + 16x - 9} \stackrel{L'H}{=} \lim_{x \rightarrow \frac{1}{2}} \frac{12x+5}{8x+16} = \frac{12(\frac{1}{2})+5}{8(\frac{1}{2})+16} = \frac{6+5}{4+16} = \frac{11}{20} \quad ? \text{ L'HOPITAL}$$

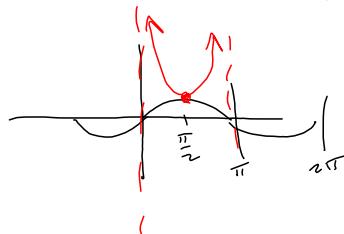
Old School

$$\frac{(3x+4)(2x-1)}{(2x+9)(2x-1)} = \frac{3x+4}{2x+9} \xrightarrow{x \rightarrow \frac{1}{2}} \frac{3(\frac{1}{2})+4}{2(\frac{1}{2})+9} = \frac{\frac{11}{2}}{10} = \boxed{\frac{11}{20}}$$

Sep 16-9:48 AM

13.  $\lim_{t \rightarrow 0} \frac{e^{2t} - 1}{\sin t}$  L'H  $\lim_{t \rightarrow 0} \frac{2e^{2t}}{\cos(t)} = \frac{2}{1} = 2$

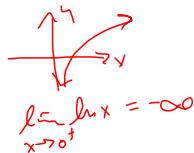
is  $\frac{0}{0}$



16.  $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{\csc \theta} = \frac{0}{1} = 0$

Sep 15 2:55 PM

19.  $\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty$



D 23.  $\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$

$$\begin{aligned} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) - \frac{1}{2}(-4x+1)^{-\frac{1}{2}}(-4)}{1} \\ &= \lim_{x \rightarrow 0} \left( \sqrt{\frac{1}{2x+1}} + \sqrt{\frac{2}{-4x+1}} \right) = \frac{1}{\sqrt{1}} + \frac{2}{\sqrt{1}} = 3 \end{aligned}$$

$$\left( \frac{\sqrt{2x+1} - \sqrt{-4x+1}}{x} \right) \left( \frac{\sqrt{2x+1} + \sqrt{-4x+1}}{\sqrt{2x+1} + \sqrt{-4x+1}} \right)$$

$$\begin{aligned} &= \frac{2x+1 - (-4x+1)}{x(\sqrt{2x+1} + \sqrt{-4x+1})} - \frac{6x}{x(\sqrt{2x+1} + \sqrt{-4x+1})} \\ &= \frac{6}{\sqrt{2x+1} + \sqrt{-4x+1}} \xrightarrow{x \rightarrow 0} \frac{6}{\sqrt{1} + \sqrt{1}} = \frac{6}{2} = 3 \end{aligned}$$

Sep 15 2:56 PM

$$26. \lim_{x \rightarrow 0} \frac{\sinh x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cosh x - 1}{3x^2} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\sinh x}{6x} \stackrel{L'H}{=} \frac{\cosh x}{6} \boxed{\frac{1}{6}}$$

$$\frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right] = \frac{e^x - e^{-x}}{2}$$

$$(30) \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = \lim_{x \rightarrow \infty} \frac{2(\ln x)}{1} = \infty$$

$\frac{\infty}{\infty}$

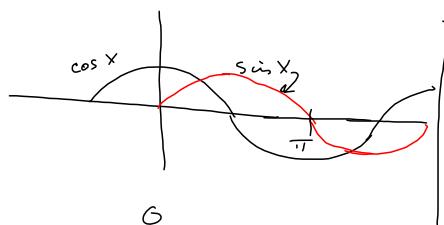
Sep 15 2:56 PM

$$(41) \lim_{x \rightarrow \infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{\cos\left(\frac{\pi}{x}\right) \cdot \left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \cos\left(\frac{\pi}{x}\right) \cdot \pi$$

$\frac{\infty \cdot 0}{1} = \frac{0}{0}$        $\boxed{\pi}$

$$\frac{\pi}{x} \xrightarrow{x \rightarrow \infty} 0$$

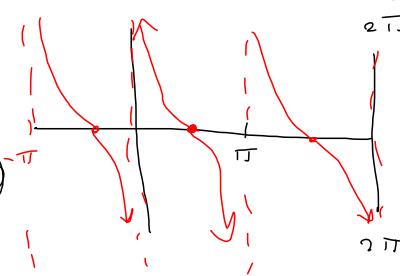
$$\sin\left(\frac{\pi}{x}\right) \xrightarrow{x \rightarrow \infty} 0$$



$$(42) \lim_{x \rightarrow 0} \cot(2x) \sin(6x)$$

from left:  $-\infty \cdot 0$   
from right:  $\infty \cdot 0$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(6x)}{\sin(2x)} \xrightarrow{0} 0 \\ &\stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{-2\sin(2x) \sin(6x) - \cos(2x)(6\cos(6x))}{2\cos(2x)} \\ &= -\frac{\cos(0)(6\cos(0))}{2\cos(0)} = \frac{6}{2} = 3 \end{aligned}$$



Sep 15 2:51 PM

$$\begin{aligned}
 & \textcircled{54} \lim_{x \rightarrow 1^+} \left[ \ln(x^5 - 1) - \ln(x^5 - 1) \right] \quad \begin{aligned} x^5 - 1 &= (x-1)(x^4 + x^3 + x^2 + x + 1) \\ x^5 - 1 &= (x-1)(x^4 + x^3 + x^2 + x + 1) \end{aligned} \\
 &= \lim_{x \rightarrow 1^+} \left[ \ln\left(\frac{x^5 - 1}{x^5 - 1}\right) \right] \\
 &= \lim_{x \rightarrow 1^+} \left( \ln\left(\frac{x^4 + x^3 + x^2 + x + 1}{x^4 + x^3 + x^2 + x + 1}\right) \right) \quad (x-1) \\
 &= \ln\left(\frac{x^4 + x^3 + x^2 + x + 1}{x^4 + x^3 + x^2 + x + 1}\right) \\
 &= \ln\left(\frac{1}{5}\right) \quad = x^5 + x^4 + x^3 + x^2 + x \\
 &\quad - x^4 - x^3 - x^2 - x - 1
 \end{aligned}$$

Sep 16-10:43 AM

$$\begin{aligned}
 & \textcircled{58} \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} \quad \left(1 + \frac{1}{x}\right)^x \xrightarrow{x \rightarrow \infty} e \\
 &= \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{2x \cdot \frac{b}{2}} \\
 &= \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{\frac{a}{2} \cdot 2b} \quad \rightarrow \left(1 + \frac{a}{x}\right)^{ab} \\
 &= \lim_{u \rightarrow \infty} \left(1 + \frac{a}{u}\right)^{u \cdot ab} \quad u = \frac{x}{2} \quad y = \left(1 + \frac{a}{x}\right)^{bx} \\
 &= e^{ab} \quad \ln y = \ln\left(\left(1 + \frac{a}{x}\right)^{bx}\right) \\
 &\text{Look for } f(x)^{g(x)} \\
 &\text{examples, which require} \\
 &y = f(x)^{g(x)} \rightarrow \\
 &\ln y = \ln(f(x)^{g(x)}) \\
 &\ln y = g(x) \ln(f(x)) \\
 &\text{want } \lim_{x \rightarrow \infty} \left( b \times \ln\left(1 + \frac{a}{x}\right) \right) \\
 &= \lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{b}x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{-\frac{a}{x^2}}{-\frac{1}{b}x} \\
 &= \lim_{x \rightarrow \infty} \left( \frac{abx}{x+a} \right) = \frac{ab}{1} = ab \\
 &\therefore \lim_{x \rightarrow \infty} \ln(y) = ab \Rightarrow \\
 &\ln(\lim_{x \rightarrow \infty} y) = ab \\
 &\lim_{x \rightarrow \infty} y = e^{ab}
 \end{aligned}$$

Sep 16-10:57 AM

$$\begin{aligned}
 1 + \frac{2}{x} &= \frac{x+2}{x} \\
 \left( \frac{-\frac{2}{x^2}}{\frac{x+2}{x}} \right) &= \frac{-\frac{2}{x^2} \cdot \frac{x}{x+2}}{-\frac{1}{5x^2}} = -\frac{2}{x(x+2)} \cdot \left(-\frac{6x^2}{1}\right) = \\
 &= \frac{-\frac{2}{x(x+2)}}{-\frac{1}{5x^2}} = \frac{abx}{x+2} \quad \text{want } x \rightarrow \infty
 \end{aligned}$$

Sep 16-11:05 AM