Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

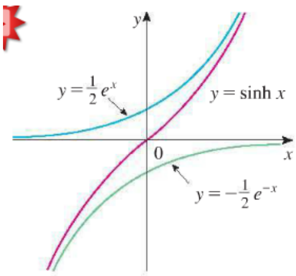


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

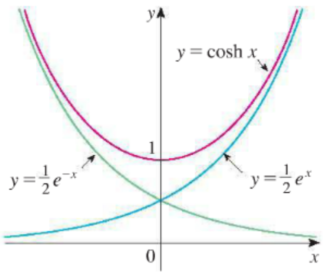


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

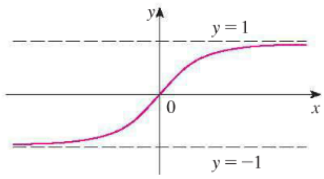
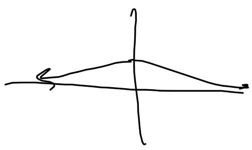


FIGURE 3
$y = \tanh x$



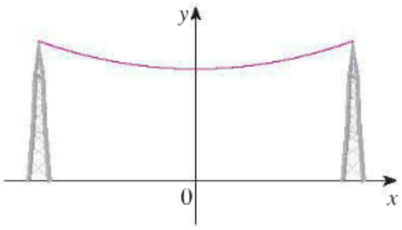


FIGURE 4
A catenary $y = c + a \cosh(x/a)$

Suspension Bridge
Power Lines

Hyperbolic Identities

$\sinh(-x) = -\sinh x$ $\qquad$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$-\tanh^2 x = \operatorname{sech}^2 x - 1$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

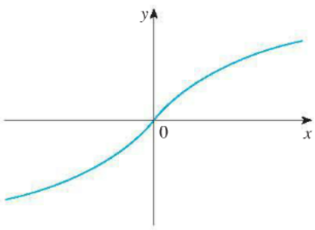


FIGURE 8 $y = \sinh^{-1} x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

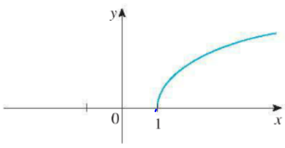


FIGURE 9 $y = \cosh^{-1} x$
domain $= [1, \infty)$ range $= [0, \infty)$

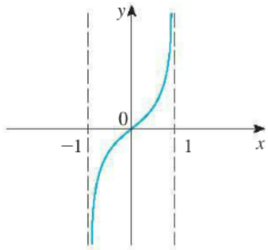


FIGURE 10 $y = \tanh^{-1} x$
domain $= (-1, 1)$ range $= \mathbb{R}$

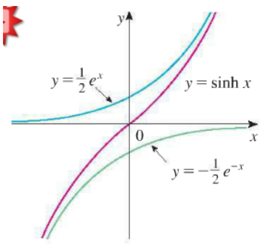


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

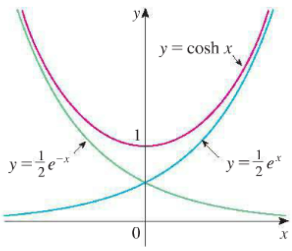


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

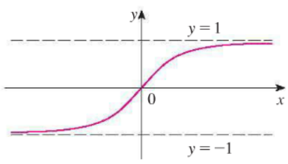


FIGURE 3
$y = \tanh x$

FACT $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

Proof
Find inverse of $\sinh x$:

$y = \sinh^{-1} x$

$\sinh(y) = x$

$\dfrac{e^y - e^{-y}}{2} = x$

$e^y - e^{-y} = 2x$

$e^y - 2x - e^{-y} = 0$

$\boxed{(e^y)^2 - 2x e^y - 1 = 0}$

Equation is quadratic in form, with variable

$\boxed{u^2 - 2xu - 1 = 0}$ → Let $u = e^y$

$au^2 + bu + c = 0$

$a = 1,\ b = -2x,\ c = 1$

$b^2 - 4ac = (-2x)^2 - 4(1)(-1)$

$= 4x^2 + 4$

$u = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2x \pm \sqrt{4x^2+4}}{2(1)} = \dfrac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^y = u = x \pm \sqrt{x^2+1}$ $x < \sqrt{x^2+1}$

$0 < e^y = x \pm \sqrt{x^2+1}$ $\sqrt{x^2+1} > \sqrt{x^2} = |x| \geq x$

$e^y = x + \sqrt{x^2+1}$

$y = \ln\left(x + \sqrt{x^2+1}\right) = \sinh^{-1} x$

FACT $\frac{d}{dx}\left[\sinh^{-1} x\right] = \frac{1}{\sqrt{x^2+1}}$

Book Proof on pg 466 is slick, I'll also do it directly

My Bludgeon:

$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ ...

$y' = \frac{1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{x + \sqrt{x^2+1}}$

⋮

$= \frac{1}{\sqrt{x^2+1}}$

Book

$y = \sinh^{-1} x$

$\frac{d}{dx}\left[\sinh y = x\right]$

$(\cosh y) \cdot y' = 1$

$y' = \frac{1}{\cosh(y)}$

$\frac{dy}{dx} = \frac{1}{\cosh(y)}$

$= \frac{1}{\sqrt{\sinh^2 y + 1}}$

$= \frac{1}{\sqrt{x^2+1}}$

Book way is a nice way to quickly re-invent what you need, rather than memorizing a ton of stuff

$\cosh^2 u - \sinh^2 u = 1$

$\cosh^2 u = \sinh^2 u + 1$

$\cosh u = \pm\sqrt{\sinh^2 u + 1}$

$\cosh u = \sqrt{\sinh^2 u + 1}$

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

6.7 Exercises

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$
2. (a) $\tanh 0$ (b) $\tanh 1$
3. (a) $\sinh(\ln 2)$ (b) $\sinh 2$
4. (a) $\cosh 3$ (b) $\cosh(\ln 3)$
5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$
6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)
8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
13. $\coth^2 x - 1 = \operatorname{csch}^2 x$
14. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
15. $\sinh 2x = 2 \sinh x \cosh x$
16. $\cosh 2x = \cosh^2 x + \sinh^2 x$
17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$
18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$
19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic function at x.
21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperboli functions at x.
22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.
23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$
(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$
(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$
(g) $\lim_{x\to0^+} \coth x$ (h) $\lim_{x\to0^-} \coth x$
(i) $\lim_{x\to-\infty} \operatorname{csch} x$
24. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.
25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1} x$ and then using Exercise 9 and Example 1(a) with x replaced by y.
26. Prove Equation 4.
27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.
28. For each of the following functions (i) give a definition like those in $\boxed{2}$, (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$
29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–45 Find the derivative. Simplify where possible.

30. $f(x) = \tanh(1 + e^{2x})$

31. $f(x) = x \sinh x - \cosh x$

32. $g(x) = \cosh(\ln x)$

33. $h(x) = \ln(\cosh x)$

34. $y = x \coth(1 + x^2)$

35. $y = e^{\cosh 3x}$

36. $f(t) = \operatorname{csch} t\,(1 - \ln \operatorname{csch} t)$

37. $f(t) = \operatorname{sech}^2(e^t)$

38. $y = \sinh(\cosh x)$

39. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x}$

40. $y = \sinh^{-1}(\tan x)$

41. $y = \cosh^{-1}\sqrt{x}$

42. $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

43. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

44. $y = \operatorname{sech}^{-1}(e^{-x})$

45. $y = \coth^{-1}(\sec x)$

46. Show that $\dfrac{d}{dx}\sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2}e^{x/2}$

47. Show that $\dfrac{d}{dx}\arctan(\tanh x) = \operatorname{sech} 2x$.

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

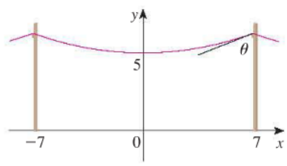
where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

50. A flexible cable always hangs in the shape of a catenary $y = c + a\cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a\cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20\cosh(x/20) - 15$, where x and y are measured in meters.
(a) Find the slope of this curve where it meets the right pole.
(b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. A cable with linear density $\rho = 2$ kg/m is strung from the tops of two poles that are 200 m apart.
(a) Use Exercise 52 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?
(b) If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

54. Evaluate $\displaystyle\lim_{x\to\infty}\frac{\sinh x}{e^x}$.

55. (a) Show that any function of the form

$$y = A\sinh mx + B\cosh mx$$

satisfies the differential equation $y'' = m^2y$.
(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

56. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

57. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

58. Investigate the family of functions

$$f_n(x) = \tanh(n \sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

59–67 Evaluate the integral.

59. $\displaystyle\int \sinh x \cosh^2 x\,dx$

60. $\displaystyle\int \sinh(1 + 4x)\,dx$

1–6 Find the numerical value of each expression.

1. (a) sinh 0 (b) cosh 0

2. (a) tanh 0 (b) tanh 1

3. (a) sinh(ln 2) (b) sinh 2

4. (a) cosh 3 (b) cosh(ln 3)

1 a $\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$

4 b $\cosh(\ln(3)) = \frac{e^{\ln(3)} + e^{-\ln(3)}}{2} = \frac{e^{\ln(3)} + e^{\ln(3^{-1})}}{2} = \frac{3+3^{-1}}{2}$

$= \frac{3+\frac{1}{3}}{2} = \frac{\frac{9}{3}+\frac{1}{3}}{2} = \frac{\frac{10}{3}}{2} = \frac{5}{3}$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

e

$$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -1\frac{(e^{x} - e^{-x})}{2} = -\sinh(x)$$

10. $\cosh x - \sinh x = e^{-x}$

$a^2 - b^2 = (a-b)(a+b)$

15. $\sinh 2x = 2 \sinh x \cosh x$

$$\frac{e^{2x} - e^{-2x}}{2} = \frac{(e^{x})^2 - (e^{-x})^2}{2} = \frac{(e^{x} - e^{-x})(e^{x} + e^{-x})}{2}$$

$$= \frac{2(e^{x} - e^{-x})(e^{x} + e^{-x})}{2 \cdot 2} = 2\sinh(x)\cosh(x)$$

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x = \sinh^2 x + 1$

$\tanh^2 x = 1 - \operatorname{sech}^2 x = \left(\frac{12}{13}\right)^2 = \frac{144}{169}$

$-\operatorname{sech}^2 x = -1 + \frac{144}{169} = \frac{-169}{169} + \frac{144}{169} = -\frac{25}{169}$

$\operatorname{sech}^2 x = \frac{25}{169}$

$\operatorname{sech} x = |\operatorname{sech} x| = \sqrt{\frac{25}{169}} = \frac{5}{13}$

$\cosh x = \frac{13}{5}$

To get $\sinh x = \cosh x \tanh x$ from $\tanh x = \frac{\sinh x}{\cosh x}$

$= \frac{13}{5} \cdot \frac{12}{13} = \frac{12}{5}$

coth, csch, sech are all fine.
Don't need to do 'em

23. Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$

(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$

(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$

(g) $\lim_{x\to0^+} \coth x$ (h) $\lim_{x\to0^-} \coth x$

(i) $\lim_{x\to-\infty} \operatorname{csch} x$

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{\cosh x}{\sinh x}$$

$$\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty} \frac{e^x(1+e^{-2x})}{e^x(1-e^{-2x})} =$$

$$\lim_{x\to\infty} \frac{1+e^{-2y}}{1-e^{-2x}} = 1$$

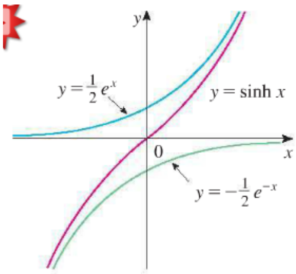


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

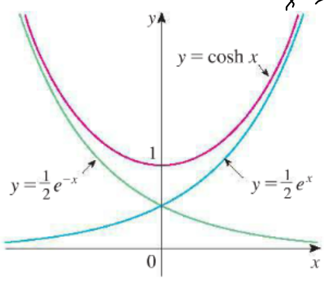


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

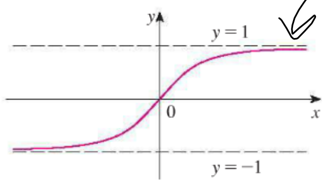


FIGURE 3
$y = \tanh x$

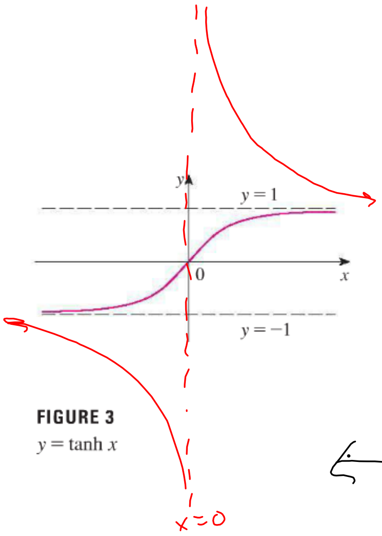


FIGURE 3
$y = \tanh x$

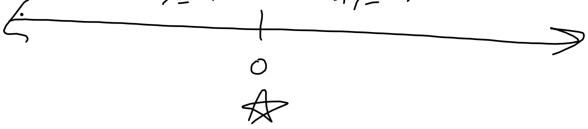
$\coth x = \frac{1}{\tanh x}$

$$\boxed{\lim_{x \to 0^-} \coth x = -\infty}$$

$\frac{e^x + e^{-x}}{e^x - e^{-x}}$

$\to 0$ so fraction blows up!

+/- ? +/- ?



30–45 Find the derivative. Simplify where possible.

$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$

30. $f(x) = \tanh(1 + e^{2x})$

$$\boxed{f'(x) = \operatorname{sech}^2(1+e^{2x}) \cdot 2e^{2x}}$$

$\frac{d}{dx}[\cosh x] = +\sinh x$

33. $h(x) = \ln(\cosh x)$

$$\boxed{h'(x) = \frac{\sinh x}{\cosh x}} = \tanh x$$

$\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$

36. $h(t) = \underbrace{\operatorname{csch} t}_{f}\underbrace{(1 - \ln \operatorname{csch} t)}_{g}$

$(fg)' = f'g + fg'$

$$\underbrace{(-\operatorname{csch} t \coth t)}_{f'}\underbrace{(1 - \ln(\operatorname{csch} t))}_{g} + \underbrace{(\operatorname{csch} t)}_{f}\underbrace{\left(-\left(\frac{-\operatorname{csch} t \coth t}{\operatorname{csch} t}\right)\right)}_{g'}$$

42. $y = x \tanh^{-1} x + \ln \sqrt{1 - x^2}$

$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2}$

$$1 \tanh^{-1} x + x \cdot \frac{1}{1-x^2} + \frac{-\frac{x}{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$$

"inverse hyperbolic tangent"

$$= \tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}$$

$$= \boxed{\tanh^{-1} x} \;!$$

$\frac{d}{dx}[\ln(f)] = \frac{f'}{f}$

$$\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

54. Evaluate $\lim_{x\to\infty} \frac{\sinh x}{e^x}$.

6.8
L'HOPITAL'S RULE
helps
IMMENSELY!

is of the form

$\frac{\infty}{\infty} \neq 1$ (maybe)

$$\frac{\sinh x}{e^x} = \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \frac{1}{e^x}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x - e^{-x}}{2e^x} = \frac{e^x}{2e^x} - \frac{e^{-x}}{2e^x}$$

$$= \frac{1}{2} - \frac{1}{2e^x} = \frac{1}{2} - \frac{1}{2}e^{-x} \xrightarrow{x \to \infty} \boxed{\frac{1}{2}}$$

55. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

$y'' = 9y = m^2 y = 3^2 y$ $\boxed{y = A \sinh(3x) + B\cosh(3x)}$

$y(0) = -4 = A \cdot 0 + B \cdot 1 \Rightarrow B = -4$

$y'(0) = 3A \underbrace{\cosh(3x)}_{1} + 3(-4) \cdot 0 = 6 \Rightarrow A = 2$

$y' = mA \cosh(mx) + mB \sinh(mx)$

$y'' = m^2 A \sinh(mx) + m^2 B \cosh(mx) = m^2 y$ ✓ Yes.

$y'' = m^2 y$? Yes. See?

$y'' = m^2 (A \sinh(mx) + B \cosh(mx))$

$= m^2 y$