

## 6.6 Inverse Trigonometric Functions

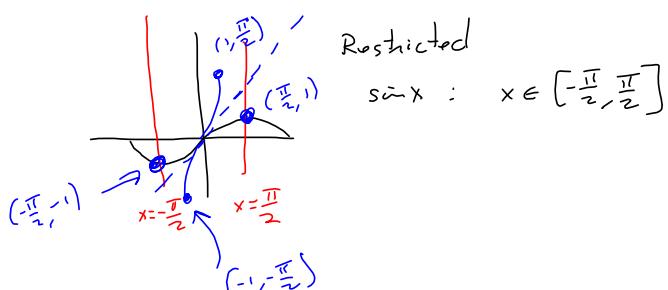
1

$$\sin^{-1}x = y \iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

2

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \text{ for } -1 \leq x \leq 1$$



3

$$\frac{d}{dx} (\underbrace{\sin^{-1} x}_{\theta}) = \frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1$$



$$\begin{aligned}
 y &= \sin^{-1} x \\
 \sin y &= x \\
 \cos(y) \cdot y' &= 1 \\
 y' &= \frac{1}{\cos(y)} = \sec(y) = \sec(\sin^{-1} x) \\
 &= \frac{1}{\sqrt{1-x^2}}
 \end{aligned}$$

Don't memorize,  
But remember this  
idea

**4**

$$\cos^{-1}x = y \iff \cos y = x \text{ and } 0 \leq y \leq \pi$$

**5**

$$\cos^{-1}(\cos x) = x \text{ for } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1}x) = x \text{ for } -1 \leq x \leq 1$$

**6**

$$\frac{d}{dx} (\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad -1 < x < 1$$

7

$$\tan^{-1}x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

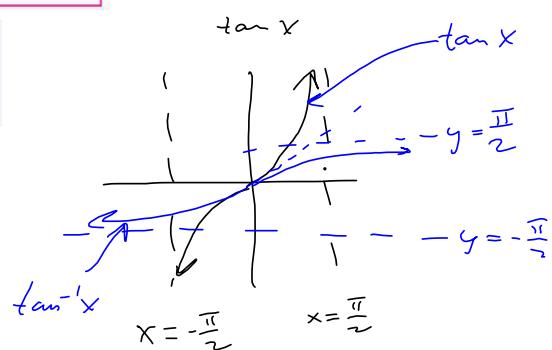
Restricted

8

$$\lim_{x \rightarrow \infty} \tan^{-1}x = \frac{\pi}{2} \quad \lim_{x \rightarrow -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

9

$$\frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2}$$



- 10**  $y = \csc^{-1}x$  ( $|x| \geq 1$ )  $\iff \csc y = x$  and  $y \in (0, \pi/2] \cup (\pi, 3\pi/2]$
- $y = \sec^{-1}x$  ( $|x| \geq 1$ )  $\iff \sec y = x$  and  $y \in [0, \pi/2) \cup [\pi, 3\pi/2)$
- $y = \cot^{-1}x$  ( $x \in \mathbb{R}$ )  $\iff \cot y = x$  and  $y \in (0, \pi)$

*Bleah***11 Table of Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

12

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$$

13

$$\int \frac{1}{x^2+1} dx = \tan^{-1}x + C$$

14

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

#14 - A trick that's used  
in homework

$$\frac{d}{dx} \tan^{-1}x = \frac{1}{x^2+1}$$

$$\int \frac{dx}{x^2+1} = \tan^{-1}x + C$$

use this to get #14

$$\frac{1}{x^2+a^2} = \frac{1}{a^2\left(\frac{x^2}{a^2}+1\right)} = \frac{1}{a^2} \cdot \frac{1}{\frac{x^2}{a^2}+1}$$

Let  $u = \frac{x}{a}$ , then we have

$$\begin{aligned} &= \frac{1}{a^2} \cdot \frac{1}{u^2+1} \\ \frac{1}{a^2} \int \frac{dx}{u^2+1} &= \frac{1}{a^2} \int \frac{adu}{u^2+1} = \frac{1}{a} \int \frac{du}{u^2+1} \\ &= \frac{1}{a} \tan^{-1}u + C = \frac{1}{a} \tan\left(\frac{x}{a}\right) + C \end{aligned}$$

$\boxed{14}$

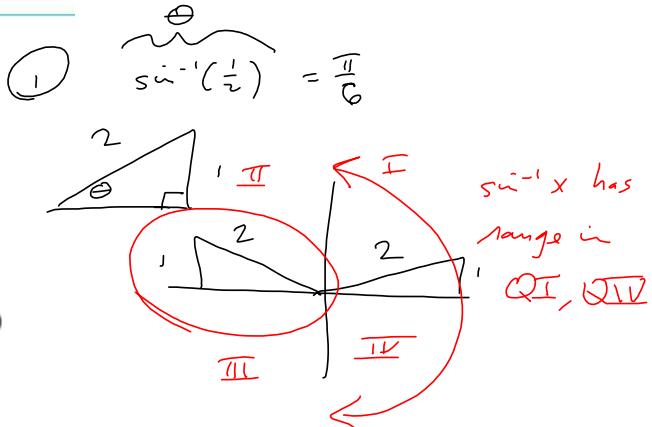
$u = \frac{x}{a} = \frac{1}{a}x \Rightarrow$   
 $du = \frac{1}{a}dx \Rightarrow$   
 $adu = dx$

## 6.6 Exercises

1-10 Find the exact value of each expression.

1. (a)  $\sin^{-1}(0.5)$

(b)  $\cos^{-1}(-1)$



2. (a)  $\tan^{-1}\sqrt{3}$

(b)  $\sec^{-1} 2$

3. (a)  $\csc^{-1}\sqrt{2}$

(b)  $\sin^{-1}(1/\sqrt{2})$

4. (a)  $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$

(b)  $\arcsin 1$

5. (a)  $\tan(\arctan 10)$

(b)  $\sin^{-1}(\sin(7\pi/3))$

6. (a)  $\tan^{-1}(\tan 3\pi/4) \times$

(b)  $\cos(\arcsin \frac{1}{2})$

7.  $\tan(\sin^{-1}(\frac{2}{3}))$

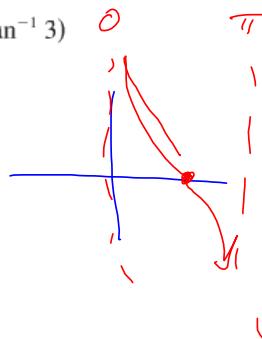
8.  $\csc(\arccos \frac{3}{5})$

9.  $\sin(2 \tan^{-1} \sqrt{2})$

10.  $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

4.  $\cot^{-1}(-\sqrt{3})$   
ref. angle  $\frac{\pi}{6}$   
 $180^\circ - 30^\circ = 150^\circ$   
 $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

\*1



7.  $\tan(\sin^{-1}(\frac{2}{3})) = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

3  
2  
 $\sqrt{3^2 - 2^2}$   
 $= \sqrt{5}$

10.  $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

$= \cos(\tan^{-1} 2) \cos(\tan^{-1} 3)$

$- \sin(\tan^{-1} 2) \sin(\tan^{-1} 3) = \text{ofc}$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sqrt{5}$   
2  
 $\tan^{-1} 2$

$\sqrt{10}$   
3  
 $\tan^{-1} 3$

11. Prove that  $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ .

11. "Proof by picture"

12-14 Simplify the expression.

12.  $\tan(\sin^{-1} x)$



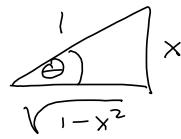
13.  $\sin(\tan^{-1} x)$

14.  $\cos(2 \tan^{-1} x)$

14.  $\cos(2\theta)$

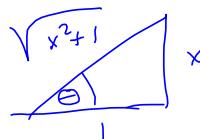
$$= \cos^2 \theta - \sin^2 \theta$$

Let  $\theta = \sin^{-1} x$



$$\cos(\theta) = \frac{\sqrt{1-x^2}}{1}$$

$$\cos(2 \tan^{-1} x) = \cos^2(\tan^{-1} x) - \sin^2(\tan^{-1} x)$$



$$\begin{aligned} & \left(\frac{1}{\sqrt{x^2+1}}\right)^2 - \left(\frac{x}{\sqrt{x^2+1}}\right)^2 \\ &= \frac{1}{x^2+1} - \frac{x^2}{x^2+1} = \boxed{\frac{1-x^2}{x^2+1}} \end{aligned}$$

17. Prove Formula 6 for the derivative of  $\cos^{-1}x$  by the same method as for Formula 3.

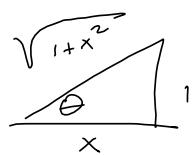
18. (a) Prove that  $\sin^{-1}x + \cos^{-1}x = \pi/2$ .  
 (b) Use part (a) to prove Formula 6.

19. Prove that  $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$ .

20. Prove that  $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$ .

21. Prove that  $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$ .

(19) Done for  $\frac{d}{dx}[\sin^{-1}x]$  in theory section.



$$\begin{aligned} y &= \cot^{-1}x \\ (\cot y &= x) \leftarrow \frac{d}{dx} \\ (-\csc^2 y) \cdot y' &= 1 \end{aligned}$$

$$\begin{aligned} y' &= -\frac{1}{\csc^2 y} = -\sin^2 y = -\sin^2(\cot^{-1}x) \\ &= -\left(\frac{1}{\sqrt{x^2+1}}\right)^2 \\ &= -\frac{1}{x^2+1} \end{aligned}$$

22-35 Find the derivative of the function. Simplify where possible.

$$22. y = \tan^{-1}(x^2)$$

$$23. y = (\tan^{-1}x)^2$$

$$25. y = \sin^{-1}(2x + 1)$$

$$27. y = x \sin^{-1}x + \sqrt{1 - x^2}$$

$$29. y = \cos^{-1}(e^{2x})$$

$$31. y = \arctan(\cos \theta)$$

$$33. h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$$

$$34. y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$$

$$35. y = \arccos\left(\frac{b + a \cos x}{a + b \cos x}\right), \quad 0 \leq x \leq \pi, \quad a > b > 0$$

$$24. y = \cos^{-1}(\sin^{-1}t)$$

$$26. g(x) = \sqrt{x^2 - 1} \sec^{-1}x$$

$$28. F(\theta) = \arcsin \sqrt{\sin \theta}$$

$$30. y = \arctan \sqrt{\frac{1-x}{1+x}}$$

$$32. y = \tan^{-1}(x - \sqrt{1+x^2})$$

**11 Table of Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$(22) \quad y = \tan^{-1}(x^2)$$

$$= \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{x^4+1}$$

$$(23) \quad y = (\tan^{-1}x)^2$$

$$\Rightarrow y' = 2(\tan^{-1}x)^1 \left( \frac{1}{x^2+1} \right)$$

$$(28) \quad \arcsin \sqrt{\sin \theta} \\ = \sin^{-1} \left( \sqrt{\sin \theta} \right) = \frac{1}{\sqrt{1 - (\sqrt{\sin \theta})^2}} \cdot \frac{1}{2} (\sin \theta)^{-\frac{1}{2}} \cos \theta \\ = \frac{\cos \theta}{2\sqrt{\sin \theta} \sqrt{1 - \sin \theta}}$$

$$(32) \quad y = \tan^{-1} \left( x - \sqrt{x^2+1} \right) \Rightarrow \\ y' = \frac{1}{(x - \sqrt{x^2+1})^2 + 1} \cdot \left( 1 - \frac{1}{2} (x^2+1)^{-\frac{1}{2}} (2x) \right)$$

**36–37** Find the derivative of the function. Find the domains of the function and its derivative.

36.  $f(x) = \arcsin(e^x)$



37.  $g(x) = \cos^{-1}(3 - 2x)$

$$\frac{d}{dx} \left[ \arcsin x \right] = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned} & \frac{d}{dx} \left[ \arcsin(f(x)) \right] \\ &= \frac{1}{1 - (f(x))^2} \cdot f'(x) \end{aligned}$$

$$\begin{aligned} & \frac{\frac{d}{dx} \left[ \arcsin(e^x) \right]}{\sqrt{1 - (e^x)^2} \cdot e^x} \\ &= \frac{e^x}{\sqrt{1 - e^{2x}}} \end{aligned}$$

**59–70** Evaluate the integral.

59.  $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$

61.  $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$

63.  $\int \frac{1+x}{1+x^2} dx$

65.  $\int \frac{dx}{\sqrt{1-x^2} \sin^{-1}x}$

67.  $\int \frac{t^2}{\sqrt{1-t^6}} dt$

69.  $\int \frac{dx}{\sqrt{x}(1+x)}$

60.  $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

62.  $\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2}$

64.  $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$

66.  $\int \frac{1}{x\sqrt{x^2-4}} dx$

68.  $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$

70.  $\int \frac{x}{1+x^4} dx$

12

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$$

13

$$\int \frac{1}{x^2+1} dx = \tan^{-1}x + C$$

14

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

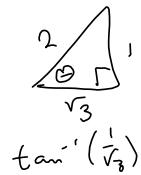
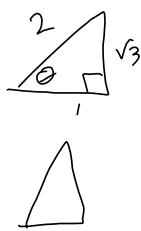
**11 Table of Derivatives of Inverse Trigonometric Functions**

$$\begin{aligned} \frac{d}{dx} (\sin^{-1}x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} (\csc^{-1}x) &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx} (\cos^{-1}x) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} (\sec^{-1}x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx} (\tan^{-1}x) &= \frac{1}{1+x^2} & \frac{d}{dx} (\cot^{-1}x) &= -\frac{1}{1+x^2} \end{aligned}$$

$$59. \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$$

$$= 8 \tan^{-1}(x) \Big|_{1/\sqrt{3}}^{\sqrt{3}} = 8 \tan^{-1}\sqrt{3} - 8 \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$8 \left[ \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2} \right] = \boxed{\frac{4\pi}{3}}$$



$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

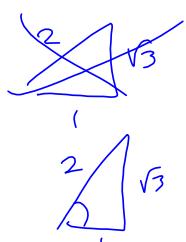
$$62. \int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2}$$

Want +  $\frac{du}{1+u^2}$   $\rightarrow 4x$

Let  $u = 4x$   $x = \frac{\sqrt{3}}{4}$   
 $du = 4dx$   $u = 4x = \sqrt{3}$   
 $dx = \frac{1}{4}du$   $x = 0 \Rightarrow u = 0$

Change limits  
 $= \int_0^{\sqrt{3}/4} \frac{\frac{1}{4}du}{1+u^2}$   
 $= \frac{1}{4} \arctan(u) \Big|_0^{\sqrt{3}/4}$

$$= \frac{1}{4} \left[ \frac{\pi}{3} - 0 \right] = \frac{\pi}{12}$$



$$\begin{aligned}
 & 65. \int \frac{dx}{\sqrt{1-x^2} \sin^{-1} x} \\
 & \text{Let } u = \sin^{-1} x \\
 & \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx \\
 & \therefore \int \left( \frac{du}{\sqrt{1-u^2}} \cdot \frac{1}{u} \right) \\
 & = \int \left( du \cdot \frac{1}{u} \right) = \int \frac{du}{u} \\
 & = \ln|u| + C \\
 & = \ln|\sin^{-1} x| + C
 \end{aligned}$$

$$\begin{aligned}
 & 66. \int \frac{1}{x\sqrt{x^2-4}} dx \\
 & \frac{x^2}{4} = \frac{x^2}{2^2} = \left(\frac{x}{2}\right)^2 \\
 & x\sqrt{x^2-4} = x\sqrt{4\left(\frac{x^2}{4}-1\right)} \\
 & = \frac{1}{x \cdot 2\sqrt{\left(\frac{x}{2}\right)^2-1}} \\
 & \text{Let } u = \frac{x}{2} \quad \text{Then } du = \frac{1}{2} dx \\
 & dx = 2du \quad x = \frac{u}{2} \\
 & \rightarrow \int \frac{2du}{2\sqrt{u^2-1}} \\
 & = \int \frac{2du}{2\sqrt{\left(\frac{u}{2}\right)\left(u^2-1\right)}} = 2 \int \frac{du}{u\sqrt{u^2-1}} \\
 & = 2 \arcsin(u) + C = 2 \underbrace{\arcsin\left(\frac{x}{2}\right)}_{\theta} + C \\
 & \text{Diagram: A right triangle with hypotenuse } \sqrt{x^2-4}, \text{ angle } \theta \text{ at the bottom-left vertex.} \\
 & = 2 \arctan\left(\frac{\sqrt{x^2-4}}{2}\right) + C
 \end{aligned}$$

$$67. \int \frac{t^2}{\sqrt{1-t^6}} dt \quad t^6 = (t^3)^2$$

$$\text{Let } u = t^3 \implies$$

$$du = 3t^2 dt$$

$$dt = \frac{du}{3t^2}$$

$$\int \frac{t^2}{\sqrt{1-u^2}} \cdot \frac{du}{3t^2}$$

$$= \frac{1}{3} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3} \arcsin(u) + C$$

$$= \frac{1}{3} \arcsin(t^3) + C$$

$$68. \int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$$

$$\text{be } \frac{u}{a} = \left(\frac{b}{a}\right)^2 \rightarrow u^2 ?$$

$$e^{4x} = e^{2x \cdot 2} = (e^{2x})^2$$

$$u = e^{2x}$$

$$du = 2e^{2x} dx$$

$$dx = \frac{du}{2e^{2x}}$$

$$\int \frac{e^{2x}}{\sqrt{1-(e^{2x})^2}} \cdot \frac{du}{2e^{2x}}$$

$$= \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C$$

$e^{2x}$

69.  $\int \frac{dx}{\sqrt{x}(1+x)}$

$u = x^{\frac{1}{2}}$

$$\begin{aligned} x = u^2 &\Rightarrow u = \pm \sqrt{x} \\ \text{Go with } u = +\sqrt{x} & \\ \text{see where } \frac{1}{2} dx \text{ leads} & \\ u = x^{\frac{1}{2}} &\end{aligned}$$

$$\begin{aligned} &\rightarrow \boxed{2x^{\frac{1}{2}} du = dx} \\ &= \frac{du}{\frac{1}{2}x^{-\frac{1}{2}}} = dx \end{aligned}$$

$$\int \frac{2x^{\frac{1}{2}} du}{x^{\frac{1}{2}}(1+u^2)} = 2 \int \frac{du}{1+u^2} = 2 \arctan(u) + C$$

$$= 2 \arctan(\sqrt{x}) + C$$

70.  $\int \frac{x}{1+x^4} dx$

$\overbrace{u = x^2}$

$$\begin{aligned} du &= 2x dx \\ \frac{du}{2x} &= dx \\ &= \int \frac{x}{1+u^2} \cdot \frac{du}{2x} \\ &= \frac{1}{2} \int \frac{du}{1+u^2} = \arctan(u) + C \\ &\text{etc.} \end{aligned}$$

