§6.5 Exponential Growth and decay

Growth rate, f', is proportional to size, f.

$y' = ky$ for some constant k.

$$\frac{y'}{y} = k$$

$$\int \frac{y'}{y}\,dx = \int k\,dx \qquad a^{b+c} = a^b a^c = a^c a^b$$

$$\ln|y| = kx + C$$

$|y| = e^{kx+C}$ Assume $y \geq 0$

$$y = e^{kx}e^{C} = e^{C}e^{kx} = \tilde{C}e^{kx}$$

$y = \tilde{C}e^{kx} \Longrightarrow$

$y' = k\tilde{C}e^{kx} \Longrightarrow$ Proportionality constant is the 'k' in the e^{kx}

It's also the RELATIVE GROWTH RATE.

$y = 100e^{.05t}$; $.05 \to k$

$y' = .05\left(100e^{.05t}\right)$

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

exponential!

$P(t) = P_0 e^{kt}$

$P(t) = 2e^{.7944t}$

k

$P_0 = 2$

P = pop as func. of
t = time, in days.

$t =$

Want $P(6) = 2e^{(.7944)(6)} \approx$ 234.9909918 $\approx \boxed{235 = P(6)}$

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 3 hours.
(c) Find the rate of growth after 3 hours.
(d) When will the population reach 10,000?

$P_0 = 100$

$P(1) = 420$

exponential

$P =$ Pop as func. of t = time, in hours.

$P_0 = 100$

$e^{2b} = (e^2)^b$

(a) $P(t) = P_0 e^{kt} = 100\, e^{kt}$

use $P(1) = 100 e^{k \cdot 1} = 420$

$100 e^k = 420$

$e^k = 4.2$

$\boxed{k = \ln(4.2)}$

$\boxed{P(t) = 100 e^{(\ln(4.2))t}}$

$= 100 \left(e^{\ln(4.2)}\right)^t$

$= 100\,(4.2)^t$

(b) $P(3) = 100 e^{(\ln(4.2))(3)}$

$\approx$ 7408.799994

$\approx \boxed{7409 \approx P(3)}$

(c) Rate of growth after 3 hours.

10632.25422

Clever: $P'(3) = k\,P(3)$

$\approx (\ln(4.2))(7408.799994)$

$P'(t) = (\ln(4.2))(100)\, e^{\ln(4.2)t}$

$P'(3) = \ln(4.2)(100)(4.2)^3 \approx \boxed{10632.25 \approx P'(3)}$

8. Strontium-90 has a half-life of 28 days.

(a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.

(b) Find the mass remaining after 40 days.

(c) How long does it take the sample to decay to a mass of 2 mg?

(d) Sketch the graph of the mass function.

(a) $P_0 e^{kt} = P(t)$, where P = Amt of radioactive Sr-90 (in mg) & t = time, (in days)

OK $\frac{1}{2}$-life eq'n

$P_0 e^{k \cdot 28} = \frac{1}{2} P_0$

$e^{28k} = \frac{1}{2}$

$28k = \ln(1/2) = -\ln(2)$

$\boxed{k = -\frac{\ln(2)}{28}}$

$\frac{1}{2}$-life version 2

$50 e^{k \cdot 28} = \frac{1}{2} \cdot 50 = 25$

$e^{28k} = \frac{1}{2}$

$$\boxed{P(t) = 50 e^{-\frac{\ln(2)}{28} t}}$$

(b) $P(40) = 50 e^{\left(-\frac{\ln(2)}{28}\right)(40)} \approx \boxed{18.57492861 \text{ mg} \approx P(40)}$

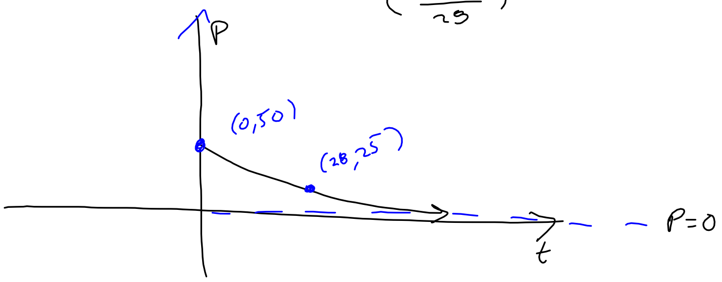
(c) $P(t) \overset{\text{SET}}{=} 2$

$50 e^{kt} = 2$

$e^{kt} = \frac{2}{50} = \frac{1}{25}$

$kt = \ln(1/25) = -\ln(25)$

$t = \frac{-\ln(25)}{k} = \frac{-\ln(25)}{\left(-\frac{\ln(2)}{28}\right)} = \frac{28 \ln(25)}{\ln(2)} \approx \boxed{130.0279733 \approx t}$



11. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

$\frac{1}{2}$-life = 5730 yrs

$P(5730) = P_0 e^{5730k} = \frac{1}{2}P_0$

$e^{5730k} = \frac{1}{2}$

$5730k = \ln(1/2) = -\ln(2)$

$k = \frac{-\ln(2)}{5730}$

$P_0 e^{kt} = .74P_0$

$e^{kt} = .74$

$kt = \ln(.74)$

$t = \frac{\ln(.74)}{k} = \frac{5730\ln(.74)}{-\ln(2)}$

$\approx$ 2489.128183 $\approx \boxed{2489 \text{ yrs} \approx AGE}$

18. (a) If \$1000 is borrowed at 8% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.

(b) Suppose \$1000 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leqslant t \leqslant 3$, graph $A(t)$ for each of the interest rates 6%, 8%, and 10% on a common screen.

$P = 1000$ dollars.
$r = .08$
$t = 3$ yrs

$A = P\left(1 + \frac{r}{m}\right)^{mt}$

(i) $m = 1$ 1259.712000
(ii) $m = 4$
(iii) $m = 12$
(iv) $m = 52$
(v) $m = 365$
(vi) $m = 365 \cdot 24$
(vii) $m = \infty$ (cnt's)

$$A := m \rightarrow 1000 \cdot \left(1 + \frac{.08}{m}\right)^{3 \cdot m}$$

$$m \rightarrow 1000\left(1 + \frac{0.08}{m}\right)$$

$A(1)$

$A(4)$

$A(12)$

$A(52)$

$A(365)$

$A(365 \cdot 24)$

$1000 \cdot \exp(.08 \cdot 3)$

```
Y1(1
          1259.712
Y1(12
       1270.237052
Y1(4
       1268.241795
```

```
       1268.241795
Y1(52
        1271.01472
```

```
        1271.01472
Y1(365
        1271.21572
Y1(365*24)
       1271.247757
1000e^(.08*3)
        1271.24915
```

\$1259.712000 Annually

\$1268.241795 Quarterly

\$1270.237067 Monthly

\$1271.014812 weekly

\$1271.215606 Daily

\$1271.233723 Hourly

\$1271.249150 Continuous