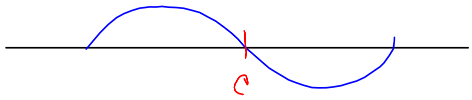
Section 5.5 - The Mean Value Theorem for Integrals

(a,b)

If f is cnt^{s} on $[a,b]$, then $\exists\ c \in [a,b] \ni$

$f(c) = f_{average}$ on $[a,b]$.

This Discussion Basically Does the Work for #23 in the textbook.



Always a $c \in (a,b)$ for this

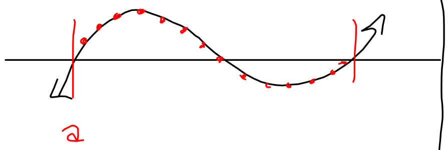
Let $F(x) = \int_a^x f(t)\,dt$. is difb^{l} on (a,b), cnt^{s} on $[a,b]$

Then by FTCI, $\exists\ c \in (a,b) \ni$

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt$$

Book says.



$$f_{AVG} \approx \frac{\sum f(x_k)}{n}$$

Recall $\Delta x = \frac{b-a}{n} \Longrightarrow$

$$n = \frac{b-a}{\Delta x}$$

$$= \sum_{k=1}^{n} f(x_k)\frac{\Delta x}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\,\Delta x \xrightarrow{n\to\infty} \frac{1}{b-a}\int_a^b f(x)\,dx = f_{AVG}.$$

1. Question Details SCalc8 5.5.001.

Find the average value f_{ave} of the function f on the given interval.

$f(x) = 3x^2 + 8x, \quad [-1, 4]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-(-1)}\int_{-1}^{4}(3x^2+8x)\,dx = \frac{1}{5}\left[x^3+4x^2\right]_{-1}^{4} =$$

$$= \frac{1}{5}\left[4^3+4(4)^2-\left((-1)^3+4(-1)^2\right)\right] = \frac{1}{5}\left[128-(3)\right] = \frac{1}{5}\left[125\right]$$

$$= \boxed{25 = f_{AVG}}$$

2. Question Details SCalc8 5.5.002.

Find the average value f_{ave} of the function f on the given interval.

$f(x) = \sqrt{x}, \quad [0, 16]$

$$f_{AVG} = \frac{1}{16}\int_0^{16} x^{\frac{1}{2}}\,dx = \frac{1}{16}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{16} = \frac{1}{16}\left[\frac{2}{3}(16)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right]$$

$$= \frac{1}{\cancel{16}}\left[\frac{2}{3}(\cancel{64})\right] = \boxed{\frac{8}{3}}$$

3. Question Details SCalc8 5.5.003.

Find the average value g_{ave} of the function g on the given interval.

Cosine is even,

$g(x) = 9\cos(x), \quad [-\pi/2, \pi/2]$

$$g_{AVG} = \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 9\cos(x)\,dx$$

$$= \frac{18}{\pi}\left[\sin(x)\right]_0^{\frac{\pi}{2}} = \boxed{\frac{18}{\pi} = g_{AVG}}$$

4. Question Details SCalc8 5.5.004.

Find the average value g_{ave} of the function g on the given interval.

$$g(t) = \frac{t}{\sqrt{5+t^2}}, \quad [2, 5]$$

$u = t^2 + 5 \Rightarrow$

$du = 2t\,dt \Rightarrow dt = \frac{du}{2t}$

$$g_{AVG} = \frac{1}{3}\int_2^5 (t^2+5)^{-\frac{1}{2}}(t\,dt) = \frac{1}{3}\int_2^5 u^{-\frac{1}{2}}\left(t\,\frac{du}{2t}\right) = \frac{1}{6}\int_2^5 u^{-\frac{1}{2}}du$$

$$= \frac{1}{6}\left[2u^{\frac{1}{2}}\right]_{2=t}^{5=t} = \frac{2}{6}\left[(t^2+5)^{\frac{1}{2}}\right]_2^5 = \frac{1}{3}\left[\sqrt{30}-\sqrt{9}\right] = \frac{1}{3}\left[\sqrt{30}-3\right]$$

$$= \boxed{\frac{\sqrt{30}}{3} - 1 = g_{AVG}}$$

5. Question Details SCalc8 5.5.007.

Find the average value h_{ave} of the function h on the given interval.

$$h(x) = 3\cos^4(x)\sin(x), \quad [0, \pi]$$

$u = \cos x \Rightarrow du = -\sin x\,dx$

$\Rightarrow dx = \frac{du}{-\sin x}$

$$h_{AVG} = \frac{3}{\pi}\int_0^{\pi} \cos^4(x)\sin(x)\,dx =$$

$$= \frac{3}{\pi}\int_{x=0}^{x=\pi} u^4 \sin(x)\cdot\frac{du}{-\sin x} = -\frac{3}{\pi}\int_{0=x}^{\pi=x} u^4\,du = \left[-\frac{3}{\pi}\cdot\frac{1}{5}u^5\right]_{0=x}^{\pi=x}$$

$$= -\frac{3}{5\pi}\left[\cos^5(x)\right]_0^{\pi} = \frac{-3}{5\pi}\left[\cos^5(\pi) - \cos^5(0)\right] = -\frac{3}{5\pi}\left[-1-(1)\right]$$

$$= \boxed{\frac{6}{5\pi} = h_{AVG}}$$

6. Question Details SCalc8 5.5.011. [3

Consider the given function and the given interval.

$$f(x) = 10\sin(x) - 5\sin(2x), \quad [0, \pi]$$

(a) Find the average value f_{ave} of f on the given interval.

f_{ave} =

(b) Find c such that $f_{ave} = f(c)$. (Round your answers to three decimal places.)

(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

(a) $$f_{avg} = \frac{1}{\pi}\int_0^{\pi}(10\sin x - 5\sin(2x))\,dx = \frac{1}{\pi}\int_0^{\pi} 10\sin x - \frac{1}{\pi}\int_0^{\pi} 5\sin(2x)\,dx$$

$u = 2x \Rightarrow$

$du = 2\,dx \Rightarrow$

$dx = \frac{du}{2}$

$$= \frac{1}{\pi}\left[-10\cos x\right]_0^{\pi} - \frac{1}{\pi}\int_{x=0}^{x=\pi} 5\sin(u)\,\frac{du}{2}$$

$$= \frac{-10}{\pi}\left[\cos(\pi) - \cos(0)\right] - \frac{5}{2\pi}\int_{x=0}^{x=\pi}\sin(u)\,du =$$

$$= -\frac{10}{\pi}\left[-1-1\right] + \frac{5}{2\pi}\left[\cos(u)\right]_{x=0}^{x=\pi} = \frac{20}{\pi} + \frac{5}{2\pi}\left[\cos(2x)\right]_0^{\pi}$$

$$= \frac{20}{\pi} + \frac{5}{2\pi}\left[\cos(2\pi) - \cos(0)\right] = \boxed{\frac{20}{\pi} = f_{AVG}}$$

(b) Find c, setting $f(x) = f_{AVG}$

$10\sin(x) - 5\sin(2x) = \frac{20}{\pi}$

$\Rightarrow 10\sin x - 5\cdot 2\sin x\cos x - \frac{20}{\pi} = 0$

Technology.

$x \approx$ 2.808120551, 1.238224521

2 spots

$\int \sin(2x)\,dx$

$= \int 2\sin x\cos x\,dx$

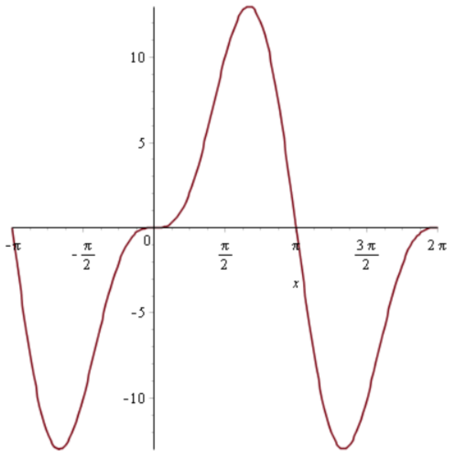
$u = \sin x$

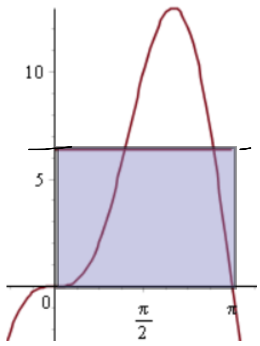
$du = \cos x\,dx$

$= \int 2u\,du = 2\cdot\frac{u^2}{2} + C$

$\boxed{= \sin^2 x + C}$

(c)





$y = \frac{20}{\pi}$

7. Question Details SCalc8 5.5.014

Find the numbers b such that the average value of $f(x) = 3 + 8x - 6x^2$ on the interval $[0, b]$ is equal to 4.

$$\frac{1}{b-0}\int_0^b (-6x^2+8x+3)\,dx = \frac{1}{b}\left[-2x^3+4x^2+3x\right]_0^b$$

$$= \frac{1}{b}\left[-2b^3+4b^2+3b\right] = -2b^2+4b+3 \overset{SET}{=} 4$$

$$\Rightarrow -2b^2+4b-1=0$$

$$\Rightarrow 2b^2-4b+1=0$$

$$b^2-4ac = (-4)^2-4(2)(1)$$

$$= 16-8 = 8$$

$$\sqrt{8} = 2\sqrt{2}$$

$$x = \frac{4\pm 2\sqrt{2}}{2(2)} = \frac{2\pm\sqrt{2}}{2}$$

b is for $b-a$.

$$\boxed{b = \frac{2\mp\sqrt{2}}{2}}$$

8. Question Details SCalc8 5.5.017.MI. [3353262

In a certain city the temperature (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 52 + 11\sin\left(\frac{\pi t}{12}\right).$$

Find the average temperature T_{ave} during the period from 9 AM to 9 PM. (Round your answer to the nearest whole number.)

$$T_{AVG} = \frac{1}{12-0}\int_0^{12}\left(52+11\sin\left(\frac{\pi t}{12}\right)\right)dt \qquad \boxed{u=\frac{\pi t}{12}} \qquad du = \frac{\pi}{12}dt$$

$$= \frac{1}{12}\left[\left[52t\right]_0^{12} + \frac{11(12)}{\pi}\int_0^{12}\sin u\,du\right] = \frac{1}{12}\left[52(12)-52(0)\right]$$

$$+\frac{11(12)}{\pi}\left[-\cos\frac{\pi t}{12}\right]_0^{12} = \frac{11(12)}{\pi}\left[-\cos(\pi)-(-\cos(0))\right] + 52$$

$$= 52 + \left(\frac{11(12)}{12\pi}\right)\left[-(-1)-(-1)\right] = 52 + \frac{22}{\pi}$$

Question Details SCalc8 5.5.018. [3353352]

The velocity v of blood that flows in a blood vessel with radius R and length ℓ at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta\ell}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity v_{ave} (with respect to r) over the interval $0 \le r \le R$. (Use *eta* for η. Use capital P for Pressure.)

Compare the average velocity v_{ave} with the maximum velocity v_{max}.

$$V_{AVG} = \frac{1}{r}\int_0^R \frac{P}{4\eta\ell}(R^2 - r^2)\,dr$$

$$= \frac{PR^2}{4r\eta\ell}\int_0^R dr - \frac{P}{4r\eta\ell}\int_0^R r^2\,dr$$

$$= B\left[R^2 r\right]_0^R - B\left[\tfrac{1}{3}r^3\right]_0^R = B\left[R^3\right] - B\left[\tfrac{1}{3}R^3\right]$$

$$= B\left[R^3 - \tfrac{1}{3}R^3\right] = \boxed{\frac{2}{3}\frac{P}{4r\eta\ell}R^3} = V_{AVG}$$

$$\boxed{V_{MAX} = \frac{P}{4r\eta\ell}R^2}$$

(when you're @ the center of the blood vessels)

10. **Question Details** SCalc8 5.5.019. [335350

The linear density ρ in a rod 7 m long is $13/\sqrt{x+9}$ kg/m, where x is measured in meters from one end of the rod. Find the average density ρ_{ave} of the rod.

$u = x + 9 \Rightarrow du = dx$

$$\rho_{AVG} = \frac{1}{7}\int_0^7 13\,(x+9)^{\frac{1}{2}}\,dx$$

$$= \frac{13}{7}\left[\frac{2}{3}(x+9)^{\frac{3}{2}}\right]_0^7 = \frac{26}{21}\left[(7+9)^{\frac{3}{2}} - (0+9)^{\frac{3}{2}}\right]$$

$$= \frac{26}{21}\left[\left((16)^{\frac{1}{2}}\right)^3 - \left(9^{\frac{1}{2}}\right)^3\right] = \frac{26}{21}\left[64 - 27\right] =$$

$$\frac{26}{21}\left[37\right] = \frac{962}{21}$$

Dec 5-1:26 PM