202 TEST 4

① $3^{\frac{1}{n}}$ converges.

Pf (1)

$$\frac{d}{dx}\left[3^{\frac{1}{x}}\right] = (\ln 3)\left(3^{\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right) < 0 \implies \text{Decreasing.}$$

$$3^{\frac{1}{n}} > 0 \implies \text{Bounded below by } 0.$$

$\implies$ Converges

Pf (2) $y = 3^{\frac{1}{n}}$

$$\ln y = \frac{1}{n}\ln 3$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n}\ln 3 = 0$$

$$\implies \lim_{n\to\infty} y = \boxed{e^0 = 1}.$$

② $3.\overline{25} = x$

$$100x = 325.\overline{25}$$

$$-\quad x = 3.\overline{25}$$

$$99x = 322$$

$$\boxed{x = \frac{322}{99}}$$

$$3 + .25 + .0025 + \cdots$$

$$= 3 + \frac{25}{100} + \frac{25}{10000} + \frac{25}{1000000} + \cdots$$

$$= 3 + 25\left(\frac{1}{10^2} + \left(\frac{1}{10^2}\right)^2 + \cdots\right)$$

$$= 3 + \frac{25}{100}\left[1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right]$$

$$= 3 + \frac{1}{4}\left[\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^{n-1}\right]$$

$$= 3 + \frac{1}{4}\left[\frac{1}{1-\frac{1}{100}}\right] = 3 + \frac{1}{4}\left(\frac{1}{\frac{99}{100}}\right)$$

$$= 3 + \frac{1}{4}\left[\frac{100}{99}\right] = 3 + \frac{25}{99} = \boxed{\frac{322}{99}}$$

202 TEST 4

(3) $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = S$

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$1 = A(n+1) + Bn$ $\quad n=-1$:

$n=0$: $\quad 1 = B(-1)$

$1 = A$

$$S_n = \sum_{n=1}^{n}\left[\frac{1}{n} - \frac{1}{n+1}\right] = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \xrightarrow{n \to \infty} \boxed{1}$$

(4) Bound on S_5's error, when $S = \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^4}$

$$\int_5^{\infty} \frac{1}{x(\ln x)^4}\,dx = \lim_{b\to\infty} \int_5^{b} (\ln x)^{-4}\left(\frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left[\frac{(\ln x)^{-3}}{-3}\right]_5^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3(\ln x)^3}\right]_5^b = \lim_{b\to\infty} \frac{-1}{3(\ln b)^3} - \frac{-1}{3(\ln 5)^3} = \frac{1}{3(\ln 5)^3} \approx \cancel{.079956}$$

$$\approx \boxed{.07995692 12}$$

$$\lim_{b\to\infty}\int_a^b (\ln x)^{-4}\left(\frac{1}{x}\,dx\right) = \dots = \frac{1}{3(\ln x)^3} < .01 \longrightarrow$$

$$3(\ln x)^3 > 100 \longrightarrow (\ln x)^3 > \frac{100}{3} \Longrightarrow \ln x > \sqrt[3]{\frac{100}{3}}$$

$$\Longrightarrow x > e^{\sqrt[3]{\frac{100}{3}}} \approx 24.99. \quad \boxed{n = 25}$$

202 TEST 4

5

(a) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2-3n-2}{n^4+n^3+11}$

5pts

$a_n < \frac{n^2}{n^4+n^3+11} < \frac{n^2}{n^4} = \frac{1}{n^2} = b_n$

$\sum b_n$ converges ($p=2$-test) $\longrightarrow$ $\boxed{\sum a_n \text{ converges}}$

5pts

(b) $\sum_{n=500}^{\infty} \frac{n^2+3n+2}{n^3-n^2-11} = \sum a_n$

$\frac{n^2+3n+2}{n^3-n^2-11} \geq$ See Bonus. Dangit.

$=$

convergence prop's $\Longrightarrow$ Both diverge ($p=1$-test)

5pts

(c) $\sum a_n = \sum_{n=500}^{\infty} \frac{n^2+3n+2}{n^3-n^2-11}$

$a_n = \frac{n^2+3n+2}{n^3-n^2-11} \geq \frac{n^2}{n^3} = \frac{1}{n} = b_n$

since $\sum b_n$ diverges, so does a_n

Didn't get the ques from I wanted for the bonus

202 TEST 4

6 (a) $\sum a_n = \sum_{n=1}^{\infty} \frac{n^2-3n-2}{n^4+n^3+11}$ $\qquad \sum b_n = \sum \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n^2-3n-2}{n^4+n^3+11} \cdot \frac{n^2}{1} = \frac{n^4-3n^3-2n^2}{n^4+n^3+11} \xrightarrow{n\to\infty} 1$$

$\rightarrow$ $\boxed{\text{Both Converge } p=2\text{-test}}$

(b) $\sum a_n = \sum_{n=500}^{\infty} \frac{n^2+3n+2}{n^3-n^2-11}$ $\qquad \sum b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n^2+3n+2}{n^3-n^2-11} \cdot \frac{n^2}{1} = \frac{n^4+3n^3+2n^2}{n^3-n^2-11} \xrightarrow{n\to\infty} 1$$

$\rightarrow$ $\boxed{\text{Both diverge, by } p=1\text{-test}}$

7 (a) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n = \sum a_n$

ROOT TEST: $\sqrt[n]{|a_n|} = \frac{2n+3}{5n+4} \xrightarrow{n\to\infty} \frac{2}{5} < 1$

$\rightarrow$ $\boxed{\text{Converges}}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{10^n} = \sum a_n$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{n+1}{10} \xrightarrow{n\to\infty} \infty > 1$

$\rightarrow$ $\boxed{\text{Diverges}}$

202 TEST 4

(c) $\sum a_n = \sum_{n=1}^{\infty} \frac{10^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \ldots = \frac{10}{n+1} \xrightarrow{n\to\infty} 0 < 1 \longrightarrow \boxed{\text{Converges}}$$

(8) $\boxed{M1}$

$$\lim_{n\to\infty}\left(\left(1-\frac{3}{n}\right)\right)^n = \lim_{n\to\infty}\left(\left(1-\frac{1}{\frac{n}{3}}\right)^{\left(\frac{n}{3}\right)(3)}\right)$$

$$= \lim_{n\to\infty}\left(\left(\left(1-\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}}\right)^3\right) = \left(e^{-1}\right)^3 = \boxed{\frac{1}{e^3}}$$

$\boxed{M2}$ $y = \left(1-\frac{3}{n}\right)^n \longrightarrow$

$$\lim_{n\to\infty}(\ln y) = \lim_{n\to\infty}\left(\ln\left(\left(1-\frac{3}{n}\right)^n\right)\right)$$

$$= \lim_{n\to\infty}\left(n \ln\left(1-\frac{3}{n}\right)\right) = \lim_{n\to\infty} \frac{\left(\ln(1-3/n)\right.}{\frac{1}{n}} \overset{L'H}{=}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{\frac{3}{n^2}}{1-\frac{3}{n}}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\left(\frac{3}{n^2}\right)\left(-\frac{1}{n^2}\right)}{1-\frac{3}{n}} = -3$$

$$\Rightarrow \lim_{n\to\infty} \ln y = -3$$

$$\Rightarrow \lim_{n\to\infty} y = e^{-3} = \boxed{\frac{1}{e^3}}$$

202 TEST 4

$\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{(n+3\sqrt{n})^3}$ How many terms to get w/in .001?

9

(a) Brute Force

$|a_1| = \frac{1}{(1+3)^3} = \frac{1}{4^3} = \frac{1}{64}$

$|a_2| = \frac{1}{(2+3\sqrt{2})^3} \approx .00411$

$|a_3| = \frac{1}{(3+3\sqrt{3})^3} \approx .0018162261$

$|a_4| = \frac{1}{(4+3\sqrt{4})^3} = (4+3\sqrt{4})^{-3} \approx .001 = \text{Error}$

This shows we only need 3 terms!

But we'd have to make the case that it's STRICTLY decreasing, blah blah blah.

$|a_5| = (5+3\sqrt{5})^{-3} \approx .000623058987\mathbf{5} < .001$

For alternating series, we NEED strictly decreasing. Easy to argue.

$n+1 + 3\sqrt{n+1} > n + 3\sqrt{n}$, since

n & $3\sqrt{n+1}$ are strictly increasing.

x^3 is increasing (strictly)

So $(n+3\sqrt{3})^3$ is strictly increasing, so

$\frac{1}{(n+3\sqrt{n})^3}$ is strictly decreasing

202 Test 4

(10) (a) $\sum a_n = \sum_{n=1}^{\infty} \frac{(2x-4)^n}{n}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x-4)^{n+1}}{n+1} \cdot \frac{n}{(2x-4)^n}\right| = \left|\frac{n}{n+1}\right| |2x-4| \xrightarrow{n\to\infty} |2x-4|$

want $|2x-4| < 1 \longrightarrow$

$-1 < 2x-4 < 1 \longrightarrow$

$3 < 2x < 5 \longrightarrow$

$\frac{3}{2} < x < \frac{5}{2}$

$\frac{\left|\frac{5}{2}-\frac{3}{2}\right|}{2} = \frac{\frac{2}{2}}{2} = \boxed{\frac{1}{2} = R}$

$x = \frac{3}{2}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

$\boxed{I = \left[\frac{3}{2}, \frac{5}{2}\right)}$

$x = \frac{5}{2}$: $\sum \frac{1}{n}$ diverges

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{(e^x-4)^n}{n}$ ... Same deal, but e^x in place of $2x \longrightarrow$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} |e^x - 4| \xrightarrow{n\to\infty} |e^x-4| \overset{\text{want}}{<} 1 \longrightarrow$

$x = \ln 3$: $\sum (-1)^n \frac{(-1)^n}{n} = \sum \frac{1}{n}$ Diverges

$-1 < e^x - 4 < 1 \longrightarrow$

$x = \ln 5$: $\sum (-1)^n \frac{1}{n}$ converges

$3 < e^x < 5 \longrightarrow$

$\boxed{I = (\ln 3, \ln 5]}$

$\ln 3 < x < \ln 5$

$\boxed{\frac{|\ln 5 - \ln 3|}{2} = R \approx .2554128119}$

202 TEST 4

11 (a) $x \ln(2x+1)$ 1st 4 nonzero terms

$\frac{d}{dx} \ln(2x+1) = \frac{2}{2x+1}$ ← Find this, then integrate.

$$\frac{2}{2x+1} = 2\left[\frac{1}{1-(-2x)}\right] = 2\left[1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots\right]$$

$$= 2\left[1 - 2x + 4x^2 - 8x^3 + \cdots\right]$$

$$= 2 - 4x + 8x^2 - 16x^3 + \cdots \longrightarrow$$

$$\ln(2x+1) = \int \left[2 - 4x + 8x^2 - 16x^3 + \cdots\right] dx + C$$

$$= \left[2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \cdots\right] + C$$

$$\ln(2(0)+1) = \left[0 + 0 + 0 + 0 + \cdots\right] + C = 0$$

$$\longrightarrow C = 0 \longrightarrow$$

$$\ln(2x+1) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \cdots \longrightarrow$$

$$x \ln(2x+1) = \boxed{2x^2 - 2x^3 + \frac{8}{3}x^4 - 4x^5} + \cdots$$

1st 4 terms

202 TEST 4

(b) $\frac{x^2}{x+3} = x^2\left[\frac{1}{3+x}\right] = x^2\left[\frac{1}{3(1+\frac{x}{3})}\right]$

$= \frac{1}{3}x^2\left[\frac{1}{1-(-\frac{x}{3})}\right] = \frac{1}{3}x^2\left[1 + (-\frac{x}{3}) + (-\frac{x}{3})^2 + (-\frac{x}{3})^3 + \cdots\right]$

$= \underbrace{\boxed{\frac{1}{3}x^2 - \frac{1}{9}x^3 + \frac{1}{27}x^4 - \frac{1}{81}x^5}}_{\text{1st 4 terms}} + \cdots$

(c) $(x^2+1)^{-\frac{1}{3}}$

$(u+1)^{-\frac{1}{3}} = \underbrace{\boxed{1 - \frac{1}{3}u + \frac{2}{9}u^2 - \frac{14}{81}u^3}}_{\text{1st 4 terms.}} + \cdots$

Fine for $(x+1)^{-\frac{1}{3}}$
WANT $(x^2+1)^{-\frac{1}{3}}$!

$\frac{(-\frac{1}{3})(-\frac{4}{3})}{2!} = \frac{\frac{4}{9}}{2} = \frac{2}{9}$

$\frac{(\frac{4}{9})(-\frac{7}{3})}{3!} = \frac{-\frac{28}{27}}{6} = -\frac{1}{6}\left(\frac{28}{27}\right) = -\frac{14}{81}$

FINAL ANSWER FINAL TEST ?

$\underbrace{\boxed{1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6}}_{\text{1st 4 nonzero terms}}$