MAT 202 Test 3, Fall, 2011

100 Points Name___________________________

This is designed as a 1-hour test, that I want to give you 2 hours to work. Some problems are more time-consuming than others. It's *nice* to get all the details, but don't waste time on details in one place and skip over main concepts in another place.

Pages 2 and 3 of this test are to help you with problems 3 and 4. Do (or start) your work on #s 3 and 4 on the sheets provided. When you're done with the test, make sure the problems are submitted in order.

1. (10 pts) Find the arc length of the curve $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$.

2. (20 pts) Find the surface area of the surface of revolution obtained by revolving $x = y^3$, from $x = 1$ to $x = 8$, about the y-axis.

3. (20 pts) A vertical plate is submerged in water and it has the shape of a semicircle of radius 4, as shown in the picture on page 2. Express the force of the water against the plate as an integral, using the picture to make your choice of coordinate axes, and set things up. Then, *evaluate* the integral. I prefer to see a symbolically precise answer in terms of Pi, fractions, and radicals (if any), rather than calculated decimal approximations. Final decimal answers should be accurate to 4 places. Use $\rho = 62.5\frac{lb}{ft^3}$ for the (constant) density of water. Since pounds represent force, the acceleration of gravity is already figured-in. Use the sheet provided for this problem.

4. (20 pts) Sketch the direction field for $y' = y - x$, using the graph paper provided. Show the graph of the solutions corresponding to $y(0) = 2$ and $y(0) = 0$ on your direction field. Sketch the Euler Method solution on the same graph, using stepsize $h = 1$, and an initial value $y(0) = 0$.

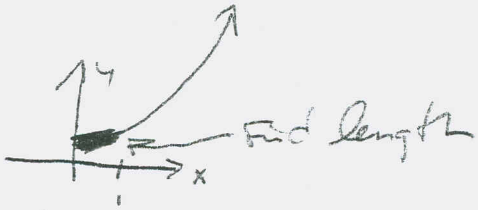
5. (20 pts) Solve the linear differential equation $y' - y = -x$, subject to an initial value $y(0) = 0$.

6. (10 pts) Use the characteristic polynomial to find the general solution to $y'' - 16y = 0$. Then find the particular solution corresponding to the boundary conditions $y'(0) = y(0) = 1$.

Bonus (10 pts) Same as #2, only revolve about the ***x*-axis**.

202 TEST 3 KEY

① $S = \int_0^1 \sqrt{1+(y')^2}\,dx$



$y = 1 + 2x^{\frac{3}{2}} \Longrightarrow y' = \frac{3}{2}\cdot 2x^{\frac{1}{2}} = 3x^{\frac{1}{2}} = 3\sqrt{x} \Longrightarrow$

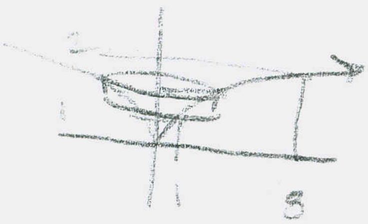
$(y')^2 = 9x \Longrightarrow$

$$S = \int_0^1 \sqrt{1+9x}\,dx = \frac{1}{9}\int_0^1 \sqrt{1+9x}\cdot 9\,dx = \frac{1}{9}\cdot\frac{2}{3}(1+9x)^{\frac{3}{2}}\Big]_0^1$$

$$= \frac{2}{27}\left[10^{\frac{3}{2}} - 1\right] = \boxed{\frac{2}{27}\left[10\sqrt{10} - 1\right]} \approx 2.268553822$$

2.268538
by calculator
integral.

② Surface area, revolving $x = y^3$ about the y-axis, from $x=1$ to $x=8$.



$x = y^3 \Rightarrow x' = 3y^2 \longrightarrow$

$$2\pi\int x\,ds = 2\pi\int_{y=1}^{y=2} y^3\sqrt{1+9y^4}\,dy \qquad \left(\frac{dx}{dy}\right)^2 = 9y^4$$

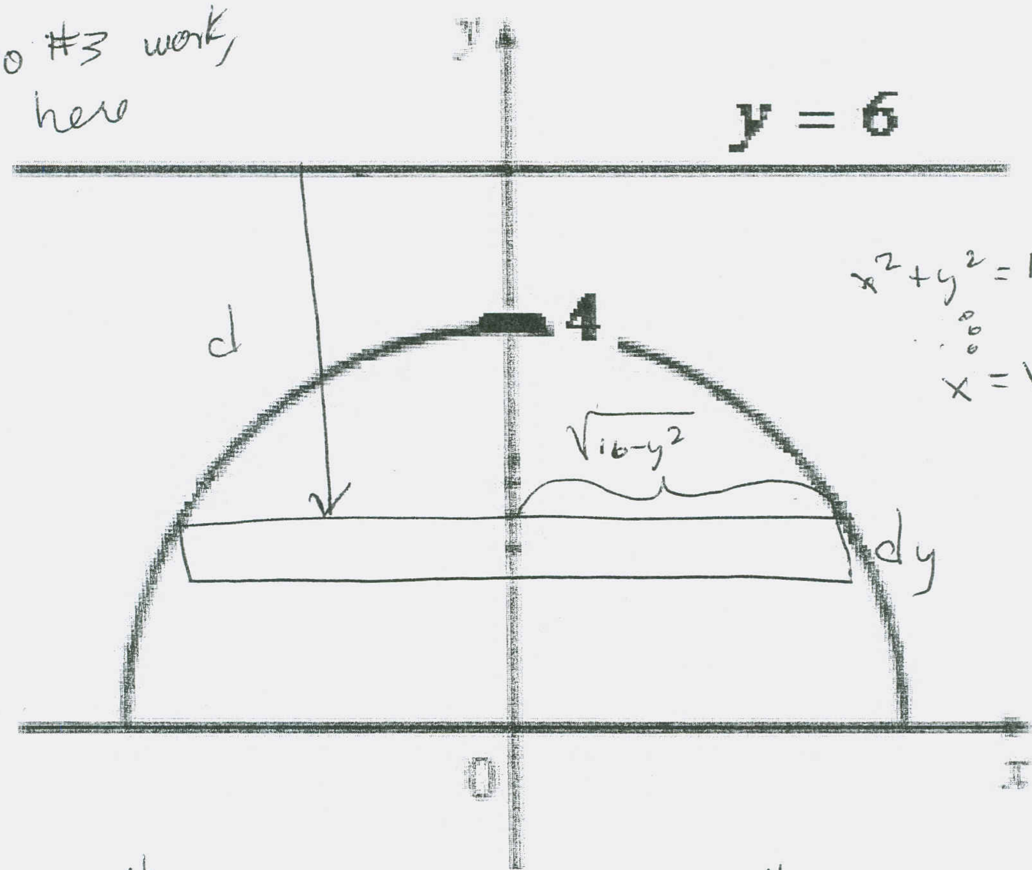
$$= 2\pi\cdot\frac{1}{36}\int_{y=1}^{y=2}\sqrt{1+9y^4}\cdot 36y^3\,dy = \frac{\pi}{18}\cdot\frac{2}{3}(1+9y^4)^{\frac{3}{2}}\Big]_1^2$$

$$= \frac{\pi}{27}\left[(1+9(16))^{\frac{3}{2}} - (1+9)^{\frac{3}{2}}\right] = \boxed{\frac{\pi}{27}\left[145\sqrt{145} - 10\sqrt{10}\right]}$$

≈ 199.4804797

Picture for #3:

Do #3 work here



$x^2+y^2=16$

$\therefore\ x=\sqrt{16-y^2}$

$$2\cdot 62.5\int_0^4 d\cdot\sqrt{16-y^2}\,dy = 125\int_0^4 (6-y)\sqrt{16-y^2}\,dy$$

$$= 125\left[\underbrace{\int_0^4 6\sqrt{16-y^2}\,dy}_{\text{Recognize it's 6 times area of quarter-circle}} - \underbrace{\int_0^4 y\sqrt{16-y^2}\,dy}_{u=16-y^2,\ du=-2y\,dy}\right]$$

$$\begin{array}{r} 750 \\ \times\ 4 \\ \hline 3000 \end{array} \qquad \begin{array}{r} 64 \\ \times\ 5 \\ \hline 320 \\ \times\ 5 \\ \hline 1600 \\ \times\ 5 \\ \hline 8000 \end{array}$$

$$= 125\left[6\cdot\frac{\pi\cdot 4^2}{4} + \frac{1}{2}\int_0^4 \sqrt{16-y^2}\cdot(-2y\,dy)\right]$$

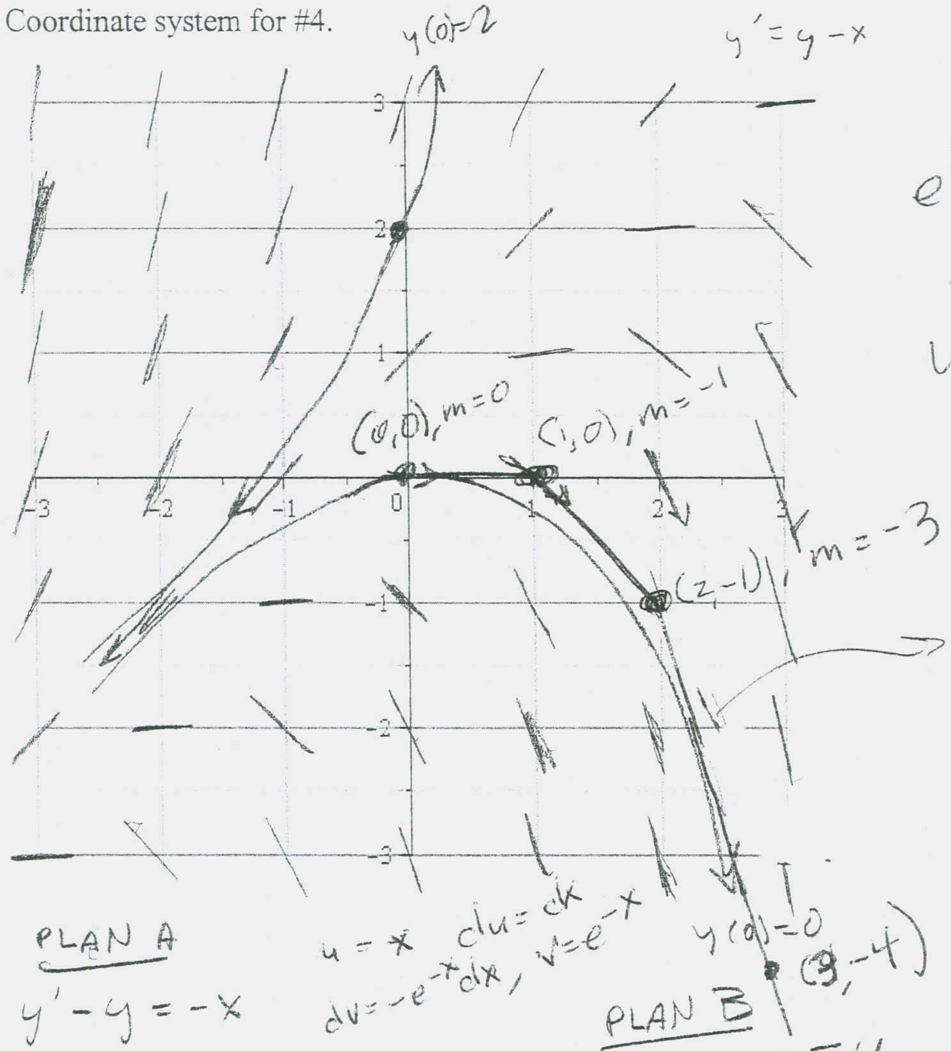
$$= 3000\pi + \frac{1}{2}\cdot\frac{2}{3}\left[(16-y^2)^{3/2}\right]_0^4 = 3000\pi + \frac{1}{3}\left[0-64\right]\cdot 125$$

$$= 3000\pi - \frac{64}{3}\cdot 125$$

$$= 3000\pi - \frac{8000}{3}$$

≈ 6758.111294 lbs

4 Coordinate system for #4.



$e^{x+C} = e^x e^C = e^C e^x = \hat{C} e^x,$ where $\hat{C} = e^C$

5 PLAN A

$y' - y = -x$

$e^{\int -dx} = e^{-x}$

$u = x \quad du = dx$

$dv = -e^{-x}dx, \quad v = e^{-x}$

$\int (e^{-x}y)' = \int -xe^{-x}$

$e^{-x}y = xe^{-x} - \int e^{-x}dx$

$e^{-x}y = xe^{-x} + e^{-x} + C$

$\Rightarrow \boxed{y = x + 1 + \hat{C}e^x}$

$y(0) = 1 + \hat{C} = 0 \Rightarrow$

$C = -1 \Rightarrow$

$\boxed{y = x + 1 - e^x}$

PLAN B

$y' = y - x \equiv u$

$\Rightarrow u' = y' - 1 \Rightarrow y' = u' + 1$

$\Rightarrow u' + 1 = u \Rightarrow$

$\Rightarrow u' = \frac{du}{dx} = u - 1$

$\int \frac{du}{u-1} = \int dx \Rightarrow \ln(u-1) = x + C$

$\Rightarrow u - 1 = e^{x+C} \Rightarrow y - x - 1 = Ke^x$

$y(0) = 0 \Rightarrow -1 = K \Rightarrow$

$\boxed{y = x + 1 - e^x}$

(6)

$$y'' - 16y = 0$$

$$D^2 - 16 = 0$$

$$D = \pm 4$$

$$\boxed{y = C_1 e^{4t} + C_2 e^{-4t}}$$

$y(0) = C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$

$y'(0) = 4C_1 - 4C_2 = 1 \Rightarrow 4(1 - C_2) - 4C_2 = 1$

$\Rightarrow 4 - 4C_2 - 4C_2 = 1$

$\Rightarrow -8C_2 = -3$

$\Rightarrow \boxed{C_2 = \frac{3}{8}} \Rightarrow C_1 = 1 - \frac{3}{8} = \boxed{\frac{5}{8} = C_1}$

So $\boxed{y = \frac{5}{8} e^{4t} + \frac{3}{8} e^{-4t}}$

(6)

Integrals for #2 the hard way about the x-axis! ①

$$2\pi \int y \, ds = 2\pi \int_1^2 y \sqrt{1 + (3y^2)^2} \, dy = I_1$$

$u = 3y^2 \qquad du = 6y\,dy$

$y = 1 \Rightarrow u = 3$

$y = 2 \Rightarrow u = 3(4) = 12$

$$\Rightarrow I_1 = \frac{2\pi}{6} \int_1^2 \sqrt{1 + (3y^2)^2} \cdot 6y\,dy$$

$$A = \frac{\pi}{3} \int_3^{12} \sqrt{1+u^2}\,du \overset{21}{=} \frac{\pi}{3}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_3^{12}$$

$$= \frac{\pi}{3}\left[\frac{12}{2}\sqrt{1+12^2} + \frac{1}{2}\ln\left(12 + \sqrt{1+12^2}\right) - \left\{\frac{3}{2}\sqrt{1+3^2} + \frac{1}{2}\ln\left(3+\sqrt{1+3^2}\right)\right\}\right]$$

$$= \frac{\pi}{3}\left[6\sqrt{145} + \frac{1}{2}\ln\left(12+\sqrt{145}\right) - \left\{\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln\left(3+\sqrt{10}\right)\right\}\right]$$

$$= \frac{\pi}{3}\left[6\sqrt{145} + \frac{1}{2}\ln\left(12+\sqrt{145}\right) - \frac{3}{2}\sqrt{10} - \frac{1}{2}\ln\left(3+\sqrt{10}\right)\right]$$

$$= \frac{\pi}{3}\left[6\sqrt{145} - \frac{3}{2}\sqrt{10} + \frac{1}{2}\ln\left(\frac{12+\sqrt{145}}{3+\sqrt{10}}\right)\right] \approx 71.40507142$$

$$B = \frac{\pi}{3}\int_3^{12}\sqrt{1+u^2}\,du = \frac{\pi}{3}\int_\alpha^\beta \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta = \frac{\pi}{3}\int_\alpha^\beta \sec^3\theta\,d\theta$$

$$= I_2$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$u = 3 = \tan\theta \rightarrow$

$\theta = \arctan(3) = \alpha$

$u = 12 = \tan\theta \rightarrow$

$\theta = \arctan(12) = \beta$

Integrals for #2 cont'd

Let $u = \sec\theta \Rightarrow du = \sec\theta\tan\theta\,d\theta$

$dv = \sec^2\theta \Rightarrow v = \tan\theta$

$$\Rightarrow I_2 = \frac{\pi}{3}\left[[\sec\theta\tan\theta]_\alpha^\beta - \int_\alpha^\beta \tan\theta\sec\theta\tan\theta\,d\theta\right] = I_3$$

$\cancel{u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta}$

$\cancel{dv = \sec\theta\tan\theta\,d\theta \Rightarrow v = \sec\theta}$

$\cancel{\Rightarrow I_2 = \frac{\pi}{3}\left[[\sec\theta\tan\theta]_\alpha^\beta - \{\sec\theta\tan\theta\right.}$ Nope. Goes in a circle.

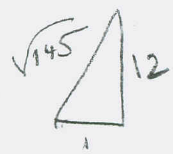
$$I_2 = \frac{\pi}{3}\left[[\sec\theta\tan\theta]_\alpha^\beta - \int_\alpha^\beta (\sec^2\theta - 1)\sec\theta\,d\theta\right]$$

$$= \frac{\pi}{3}\left[[\sec\theta\tan\theta]_\alpha^\beta - \int_\alpha^\beta \sec^3\theta\,d\theta + \int_\alpha^\beta \sec\theta\,d\theta\right]$$

$$= \frac{\pi}{3}[\sec\theta\tan\theta]_\alpha^\beta - \frac{\pi}{3}\int_\alpha^\beta \sec^3\theta\,d\theta + [\ln|\sec\theta + \tan\theta|]_\alpha^\beta$$ So

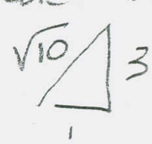
$$I_2 = \frac{\pi}{3}[\sec\theta\tan\theta]_\alpha^\beta - I_2 + [\ln|\sec\theta + \tan\theta|]\cdot\frac{\pi}{3} \Rightarrow$$

$$\Rightarrow I_2 = \frac{\pi}{6}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_\alpha^\beta$$



$\beta = \arctan(12)$

$\alpha = \arctan(3)$



$$= \frac{\pi}{6}\left[\sqrt{145}\cdot 12 + \ln|\sqrt{145} + + 12| - (\sqrt{10}\cdot 3 + \ln|\sqrt{10} + 3|)\right]$$

$$\approx 71.40507142$$