MAT 202 | Test 3 – Part II
40 Points | Name KEY

1. For what values of r does the function $y = e^{rt}$ satisfy the differential equation

$$y'' + y' - 20y = 0?$$

$(D+5)(D-4)y = 0$ $\qquad$ $(r+5)(r-4) = 0$

$r = -5, 4$ $\qquad$ $r = -5, 4$

$r^2 e^{rt} + re^{rt} - 20e^{rt} = 0 \Longrightarrow$

$(r^2 + r - 20)e^{rt} = 0 \quad \ldots$

2. For what nonzero values of k does the function $y = \sin(kt)$ satisfy $y'' + 4y = 0$?

$(D^2+4)\,y = 0$ $\qquad$ $y' = k\cos kt$

$r = \pm 2i$ $\qquad$ $y'' = -k^2 \sin kt$

$-k^2 \sin kt + 4 \sin kt = 0$

$k = \pm 2$

$(-k^2+4)\sin(kt) = 0 \longrightarrow k = \pm 2$

3. Solve the differential equation $5yy' = 3x$.

$5y \frac{dy}{dx} = 3x$

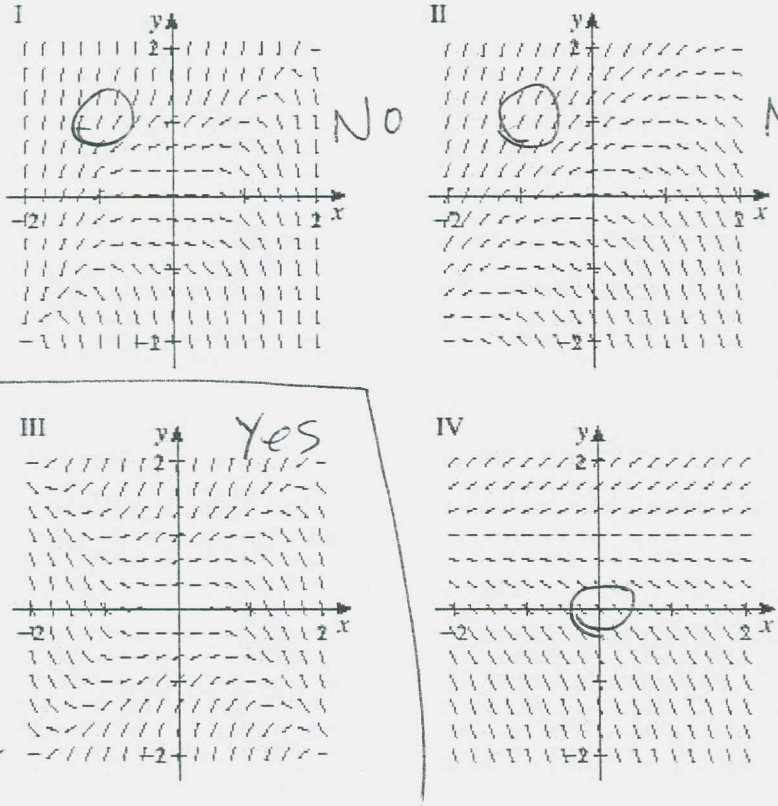
$\int 5y\,dy = \int 3x\,dx$

$\frac{5}{2}y^2 = \frac{3}{2}x^2 + C^*$

$y^2 = \frac{3}{5}x^2 + \frac{2}{5}C^*$

$y = \pm\sqrt{\frac{3}{5}x^2 + K}$, here $K = \frac{2}{5}C^*$

4. Select a direction field for the differential equation $y' = y^2 - x^2$ from a set of direction fields labeled I-IV.



I: No

II: No

III: Yes

IV: No

5. The functions $y = Ce^{2x^2}$ (for any constant C) are solutions of the differential equation $y' = 4xy$. Find the solution that satisfies the initial condition $y(1) = 1$.

$y(1) = Ce^2 = 1 \longrightarrow C = \frac{1}{e^2}$

$\Rightarrow \boxed{y = e^{-2} \cdot e^{2x^2}} = \boxed{e^{2x^2 - 2}}$

FINAL ANS

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6. Solve the differential equation $y' = \dfrac{7x^6 y}{\ln y}$

$$\ln y \frac{y'}{y} = 7x^6$$

$$\int \ln y \frac{dy}{y} = 7x^6 dx$$

$u = \ln y$

$du = \frac{y'}{y} dx = \frac{dy}{y}$

$$\Rightarrow \int u\, du = \tfrac{1}{2}u^2 = \tfrac{1}{2}(\ln y)^2 = x^7 + C$$

$$\Rightarrow (\ln y)^2 = 2x^7 + 2C$$

$$\ln y = \pm\sqrt{2x^7 + K}$$

$$y = e^{\pm\sqrt{2x^7+K}}$$

$$\boxed{y = e^{\sqrt{2x^7+K}}} \quad K = 2C$$

7. Solve the initial-value problem $x^2 \dfrac{dy}{dx} + 2xy = -\sin x$, given that $y(\pi) = 0$.

$$\frac{dy}{dx}x^2 + 2xy$$

$$= \frac{d}{dx}\left[x^2 y\right] = -\sin x$$

$$\Rightarrow x^2 y = -\int \sin x\, dx = \cos x + C$$

$$\Rightarrow y = \frac{\cos x}{x^2} + \frac{C}{x^2}$$

$$y(\pi) = \frac{\cos \pi}{\pi^2} + \frac{C}{\pi^2} = 0 \Rightarrow C = 1, \text{ so}$$

$$\boxed{y = \frac{\cos x}{x^2} + \frac{1}{x^2}}$$

8. Solve the initial value problem $\frac{dr}{dt} + 2tr = r$, given that $r(0) = 10$.

$$\frac{dr}{dt} = r - 2rt = r(1-2t)$$

$$\int \frac{dr}{r} = \int (1-2t)\,dt$$

$$\ln|r| = t - t^2 + c$$

$$|r| = e^{-t^2+t+c}$$

$$r = \pm e^{-t^2+t+c} = \pm K e^{-t^2+t}, \text{ where } K = e^c$$

$r(0) = 10$

$\Rightarrow 10 = \pm K = e^c \Rightarrow$

$K = 10 = e^c$

$c = \ln 10$ &

$r = 10e^{-t^2+t}$

Bonus – Solve the logistic equation $\frac{dp}{dt} = 0.1p\left(1 - \frac{p}{2000}\right)$, given $p(0) = 100$. (Hint: This equation is separable, and its solution involves partial fractions.)

$$\frac{dp}{dt} = \frac{1}{10} p\left(\frac{2000-p}{2000}\right) = \frac{1}{20000}\left(p(2000-p)\right)$$

$$\Rightarrow \frac{dp}{p(2000-p)} = \frac{1}{20000} \qquad \int \frac{dp}{p(2000-p)} =$$

$$\frac{1}{p(2000-p)} = \frac{A}{p} + \frac{B}{2000-p}$$

$$1 = A(2000-p) + Bp$$

$$1 = 2000A - Ap + Bp$$

$$-Ap + Bp = 0 \Rightarrow A = B$$

$$1 = 2000A \Rightarrow A = \frac{1}{2000}$$

$$= \frac{1}{2000}\left[\int \frac{dp}{p} + \frac{dp}{2000-p}\right] = \frac{1}{20000}\int dt$$

$$\Rightarrow 10\left[\ln|p| - \ln|2000-p|\right] = t$$

$$\Rightarrow \ln\left(\frac{p}{2000-p}\right) = \frac{1}{10}t$$

$$\frac{p}{2000-p} = e^{\frac{1}{10}t}$$

$$\boxed{p = \frac{2000e^{\frac{1}{10}t}}{1 - e^{\frac{1}{10}t}}}$$

$$\Rightarrow p = 2000e^{\frac{1}{10}t} - pe^{\frac{1}{10}t} \Rightarrow p\left(1 - e^{\frac{1}{10}t}\right) = 2000e^{\frac{1}{10}t}$$