MAT 202 Test 3 – Part I

40 Points Name KEY

1. Use the arc length formula to find the length of the curve $y=\sqrt{2-x^2}, 0 \le x \le 1$. (You can check by noting this is part of a circle.)

$$\int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+\frac{x^2}{2-x}}\,dx$$

$y=\sqrt{2-x^2}$

$y' = \frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x)$

$= -\frac{x}{\sqrt{2-x^2}}$

$$= \int_0^1 \sqrt{\frac{x^2-x^2+2}{2-x^2}}\,dx = \int_0^1 \frac{\sqrt{2}}{\sqrt{2-x^2}}\,dx$$

Let $x=\sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\,d\theta$

$\sqrt{2-x^2} = \sqrt{2}\cos\theta$

$x=0 \to \sin\theta = 0 \to \theta = 0$

$x=1 \to \sin\theta = \frac{1}{\sqrt{2}} \to \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{\sqrt{2}}{\sqrt{2}\cos\theta}\cdot\sqrt{2}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2}\,d\theta = \boxed{\frac{\sqrt{2}\,\pi}{4}}$$

2. Find the length of the curve $y=\ln(\sin(x))$, for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$

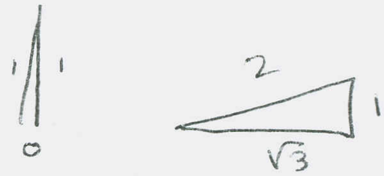
$y' = \frac{\cos x}{\sin x} = \cot x$

$\sqrt{1+(y')^2} = \sqrt{1+\cot^2 x}$

$= \csc x$

$$L = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc x\,dx$$

$$= -\ln|\csc x - \cot x| \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$



$$= -\ln|1+0| - (-\ln|2-\sqrt{3}|) = \boxed{\ln|2-\sqrt{3}|}$$

3. Find the area of the surface obtained by rotating the curve

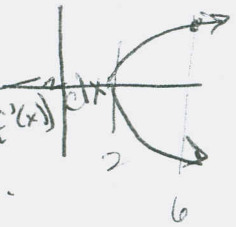
$$9x = y^2 + 18, \text{ for } 2 \le x \le 6$$

about the x-axis.

$\frac{1}{9}y^2 + 2 = 6$, $\frac{1}{9}y^2 = 4$, $y^2 = 36$, $y = 6$

$x = \frac{1}{9}y^2 + 2$

Barbara has the $2\pi\int f(x)\sqrt{1+(f'(x))^2}\,dx$ version.



$$2\pi\int_0^6 y\sqrt{1+\frac{4}{81}y^2}\,dy$$

$$= \frac{2\pi}{9}\int_0^6 y\sqrt{81+4y^2}\,dy = \frac{2\pi}{9}\int_0^6 y\sqrt{9^2+(2y)^2}\,dy$$

$\frac{dx}{dy} = \frac{2}{9}y$

$u = 2y \Rightarrow$ $y = \frac{u}{2}$, $dy = \frac{du}{2}$

$$= \frac{2\pi}{9}\int_0^{12} \frac{u}{2}\sqrt{9^2+u^2}\,\frac{du}{2}$$

$81 + 144 = 225$

$$= \frac{\pi}{18}\int_0^{12}\sqrt{9^2+u^2}\,\left(\tfrac{1}{2}\right)(2u\,du)$$

$v = 81+u^2$, $dv = 2u\,du$, $u=0,\ v=81$, $u=12,\ v=225$

$$\frac{\pi}{36}\int_{81}^{225}\sqrt{v}\,dv = \left(\frac{2}{3}\right)\left(\frac{\pi}{36}\right) v^{\frac{3}{2}}\Big|_{81}^{225}$$

$$= \frac{\pi}{(3)(18)}\left[225^{3/2} - 81^{3/2}\right]$$

$$= \frac{\pi}{(18)(3)}\left[15^3 - 9^3\right]$$

$$= \frac{\pi}{(18)(3)}\left[3375 - 729\right]$$

$$= \frac{\pi}{(18)(3)}\left[2646\right]$$

$$\boxed{= 49\pi}$$

4. Find the area of the surface obtained by rotating the curve

$$y = \frac{x^2}{4} - \frac{\ln x}{2},\ 1 \le x \le 2$$

about the y-axis.

$y' = \frac{x}{2} - \frac{1}{2x}$

$(y')^2 = \frac{x^2}{4} + \frac{1}{4x^2} - \frac{1}{2}$

$1 + (y')^2 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$

$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2$

$$2\pi\int_1^2 x\cdot\left(\frac{x}{2}+\frac{1}{2x}\right)dx$$

$$= 2\pi\int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}\right)dx$$

$$= \pi\left[\frac{x^3}{3} + 1x\right]_1^2$$

$$= \pi\left[\frac{8}{3} + 2 - \frac{1}{3} - 1\right]$$

$$= \pi\left[\frac{7}{3} + \frac{3}{3}\right] = \pi\left[\frac{10}{3}\right]$$

$$\boxed{= \frac{10\pi}{3}}$$

Alternate: $y^2 = 9x - 18$

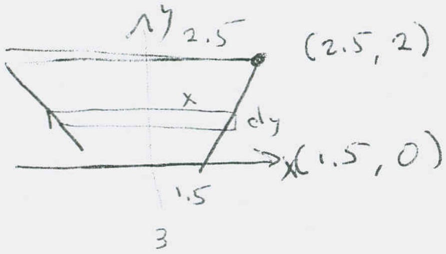
$y = \pm\sqrt{9x-18} = \pm 3\sqrt{x-2}$

$\frac{10\pi}{3}$

0.471976

Bonus – A gate in an irigation canal is constructed in the form of a trapezoid 3 ft wide at the bottom, 5 ft wide at the top, and 2 ft high. It is placed vertically in the canal, with the water extending to its top. Find the hydrostatic force on one side of the gate.

458 lbs



Depth $= 2 - y$

$m = \frac{2-0}{2.5-1.5} = \frac{2}{1}$

$y = 2(x-1.5)+0$

$y = 2x - 3$

$2x = y + 3$

$x = \frac{y+3}{2}$

$\int_0^2 \quad dy$

$\rho \; 2\int_0^2 x\,(2-y)\,dy$

$= \int_0^2 \left(\frac{y+3}{2}\right)(2-y)\,dy$

$= \rho \int_0^2 -(y+3)(y-2)\,dy$

$= -\rho \int_0^2 (y^2 + y - 6)\,dy$

$= -\rho \left[\frac{y^3}{3} + \frac{y^2}{2} - 6y\right]_0^2$

$= -\rho \left[\frac{8}{3} + 2 - 12\right]$

$= \rho \left[10 - \frac{8}{3}\right] = \frac{22}{3}\rho$

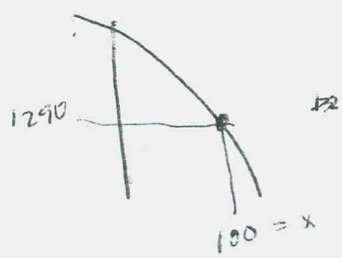
$= \frac{(22)(62.5)}{3} \approx 458.3$

≈ 458 lbs

5. The demand function for a commodity is given by

$$p = -.02x^2 - .1x + 1500$$

Find the consumer surplus when the sales level is 100



$\int_0^{100} (-.02x^2 - .1x + 1500)\,dx - 129{,}000$

$= \left[-\frac{.02}{3}x^3 - \frac{.1}{2}x^2 + 1500x\right]_0^{100} - 129{,}000$

$= 142{,}833 - 129000 = \$13{,}833$

Surface Area:

y-axis:

$2\pi \int x\, ds$

x-axis

$2\pi \int y\, ds$

Test 3 #5: (Version 2)

$p = \frac{33}{x+8}$

Consumer Surplus, when price = 20

$\int_0^{6.35} p(x)\, dx - p(6.35)(6.35)$

$\frac{33}{x+8} = 20$

$33 = 20x + 160$

$-20x = 127$

$x = \frac{-127}{20}$

$= 6.35$

$= 33 \ln(x+8) \Big]_0^{6.35} - (2.30)(6.35)$

$= 33\left[\ln(14.35) - \ln 8\right] - 14.605$

$\approx 19.28217722 - 14.605 \approx$ **\$4.68** ← Consumer Surplus

SCRATCH. TEST 3

$$\frac{d}{dx}(9x-18)^{\frac{1}{2}} = \frac{1}{2}(9x-18)^{-\frac{1}{2}}(9)$$

$$= \frac{9}{2\sqrt{9x-18}} = y' \longrightarrow$$

$$(y')^2 = \frac{81}{4(9x-18)}$$

$$\text{\&} \quad 1+(y')^2 = \frac{4(9x-18)+81}{4(9x-18)}$$

$$= \frac{36x-72+81}{LCD} = \frac{36x+9}{4(9x-18)}$$

Consumer Surplus, when price = \$20 for

$p = \frac{33}{x+8}$!? Find x:

$$20 = \frac{33}{x+8}$$

$$20x + 160 = 33$$

$$20x = -127$$

$$\boxed{x = -6.35}$$

$$\int_0^{-6.35} \frac{33}{x+8}\,dx - P \cdot x$$

$$= 33\ln(8-6.35) - 33\ln(8) - (20)(6.35)$$

$\approx$ ~~\$ -112.55~~ !?

$\approx$ \$ -179.10