

1. Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or provide a counterexample.

a. (5 pts) If  $f$  is one-to-one, with domain  $\mathbf{R}$ , then  $f^{-1}(f(6)) = 6$ .

True.  $f^{-1}(f(x)) = x \quad \forall x \in \mathcal{D}(f)$

b. (5 pts) If  $f$  is one-to-one and differentiable, with domain  $\mathbf{R}$ , then

$$(f^{-1})'(6) = \frac{1}{f'(6)}$$

False.  $(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))}$

2. The graph of  $g$  is given.

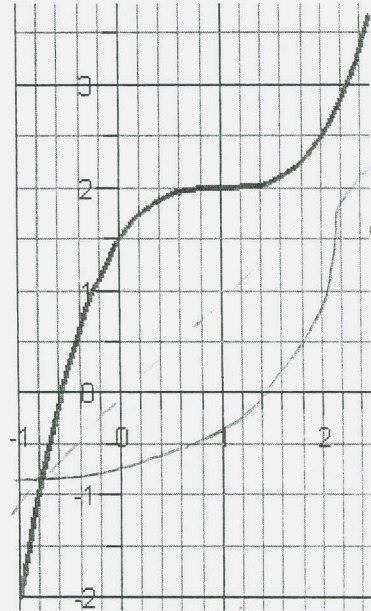
a. (5 pts) Why is  $g$  one-to-one?

It's increasing.

b. (5 pts) Estimate the value of  $g^{-1}(1)$ .

$$g^{-1}(1) \approx -0.2$$

c. (5 pts) Sketch the graph of  $g^{-1}$ .



3. Find the exact value of each of the following:

a. (5 pts)  $\ln(e^\pi) = \pi$

b. (5 pts)  $\cos(\arctan \sqrt{3}) = \frac{1}{2}$



4. Solve the following equations for  $x$ .

a. (5 pts)  $\ln(1 + e^{-x}) = 3$

$$1 + e^{-x} = e^3$$

$$e^{-x} = e^3 - 1$$

$$-x = \ln(e^3 - 1)$$

$$x = -\ln(e^3 - 1)$$

b. (5 pts)  $\ln(x+1) + \ln(x-1) = 1$

$$\ln(x^2-1) = 1$$

$$x^2-1 = e$$

$$x^2 = e+1$$

$$x = \pm \sqrt{e+1}$$

But  $x > 1$ , by domain of  $\ln(x-1)$ , so  $x = -\sqrt{e+1}$

~~X~~. Hence  $x = \sqrt{e+1}$

5. Differentiate. Do Not Simplify.

a. (4 pts)  $f(t) = t^2 \ln t$

$$f'(t) = 2t \ln t + t^2 \cdot \frac{1}{t}$$

b. (4 pts)  $g(x) = 3^{mx} \cos(nx)$

$$g'(x) = (\ln 3)(m)(3^{mx}) \cos(nx) - (3^{mx})(\sin(nx))(n)$$

c. (4 pts)  $V(t) = \arctan(\arcsin \sqrt{t})$

$$V'(t) = \frac{1}{1 + (\arcsin \sqrt{t})^2} \cdot \frac{1}{\sqrt{1 - (t)^2}} = \frac{1}{2\sqrt{t}}$$

d. (4 pts)  $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$  (Use logarithmic differentiation.)

$$\ln y = 4 \ln(x^2+1) - 3 \ln(2x+1) - 5 \ln(3x-1)$$

$$\frac{y'}{y} = 4 \left( \frac{2x}{x^2+1} \right) - 3 \left( \frac{2}{2x+1} \right) - 5 \cdot \frac{3}{3x-1}$$

$$\rightarrow y' = \left( \frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1} \right) \left( \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \right)$$

6. Cobalt-60 has a half-life of 5.24 years.

a. (4 pts) Find the mass that remains from a 100-mg sample after  $t$  years. ~~Your~~  
answer should be to the nearest 10<sup>th</sup> of a gram.

$$A_0 e^{5.24k} = \frac{1}{2} A_0$$

$$e^{5.24k} = \frac{1}{2}$$

$$A(t) = 100e^{kt}$$

$$5.24k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5.24}$$

$$\approx -0.1322799963 \approx k$$

b. (3 pts) How long would it take for the mass to decay to 1 mg? Your answer should be to the nearest year.

$$A(t) = 1$$

$$100e^{kt} = 1$$

$$e^{kt} = \frac{1}{100}$$

$$kt = \ln\left(\frac{1}{100}\right)$$

$$t = \frac{1}{k} \ln\left(\frac{1}{100}\right)$$

$$= \frac{5.24}{\ln\left(\frac{1}{2}\right)} \ln\left(\frac{1}{100}\right) \approx 34.8 \text{ yrs}$$

$$35 \text{ yrs}$$

c. (3 pts) What is the rate of decay after 10 years? What is the percentage rate of decay?

$$A'(10) = ?$$

$$A'(t) = 100ke^{kt}$$

$$A'(10) = 100ke^{10k}$$

$$= (100) \frac{\ln\left(\frac{1}{2}\right)}{5.24} e^{(10) \frac{\ln\left(\frac{1}{2}\right)}{5.24}}$$

$$\approx 3.523785301$$

$$\approx$$

$$3.523785301 \text{ g/yr}$$

% rate is

$$-13.22799963 \%$$

7. (4 pts) Find  $f'(x)$  for  $f(x) = \int_1^{\sqrt{x}} \frac{e^t}{t} dt$

$$f'(x) = \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

8. Evaluate the integral.

$$\text{a. (5 pts)} \int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx = \int_0^1 \frac{du}{1+u^2} = \arctan u \Big|_0^1$$

$$u = \sin x$$

$$du = \cos x dx$$

$$\sin(0) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\text{b. (5 pts)} \int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \arcsin u + C$$

$$u = x^2$$

$$du = 2x dx$$

$$= \frac{1}{2} \arcsin(x^2) + C$$

$$\text{c. (5 pts)} \int \ln(\cos x) \tan x dx = - \int u du = -\frac{u^2}{2} + C$$

$$u = \ln(\cos x)$$

$$du = \frac{-\sin x}{\cos x} dx$$

$$= -\tan x dx$$

$$= -\frac{(\ln(\cos x))^2}{2} + C$$

9. (5 pts) If  $f(x) = x + x^2 + e^x$ , find  $(f^{-1})'(1)$ .

$$f(x) \stackrel{?}{=} 1 \Rightarrow f'(x) = 1 + 2x + e^x$$

$$\rightarrow x = 0$$

$$f'(0) = 1 + e^0 = 2$$

$$\text{since } 0 + 0^2 + e^0 = 1$$

$$\Rightarrow (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{2}$$

$$\text{So, } f^{-1}(1) = 0$$

10. (5 pts) If  $\tanh x = \frac{3}{5}$ , find the value of the other 5 hyperbolic trigonometric functions.

This should not require a calculator.

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$= 1 - \frac{9}{25}$$

$$= \frac{25-9}{25}$$

$$= \frac{16}{25}$$

$$\Rightarrow \operatorname{sech} x = \pm \frac{4}{5}$$

$$\Rightarrow \operatorname{sech} x = \frac{4}{5}$$

$$\Rightarrow \cosh x = \frac{5}{4}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{4}{5} \sinh x = \frac{3}{5}$$

$$\Rightarrow \sinh x = \frac{3}{4}$$

$$\Rightarrow \operatorname{csch} x = \frac{4}{3}$$

$$\Rightarrow \operatorname{coth} x = \frac{5}{3}$$

11. (Bonus) (5 pts) Use mathematical induction to show that if  $f(x) = xe^x$ , then  $f^{(n)}(x) = (x+n)e^x$ .

$$\boxed{\text{PF}} \quad \text{Let } f(x) = xe^x. \text{ Then } f'(x) = f^{(1)}(x) = e^x + xe^x$$

$$= (1+x)e^x. \quad \text{f} \quad f^{(k)}(x) = (k+x)e^x = (x+k)e^x.$$

$$f^{(k+1)}(x) = \left( f^{(k)}(x) \right)' = \frac{d}{dx} \left[ (k+x)e^x \right] = e^x + (k+x)e^x$$

$$= (1+k+x)e^x = (k+1+x)e^x = (x+(k+1))e^x \quad \blacksquare$$