MAT ~~121~~ 202 FALL '09 Test 1

100 Points Name KEY

1. Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or provide a counterexample.

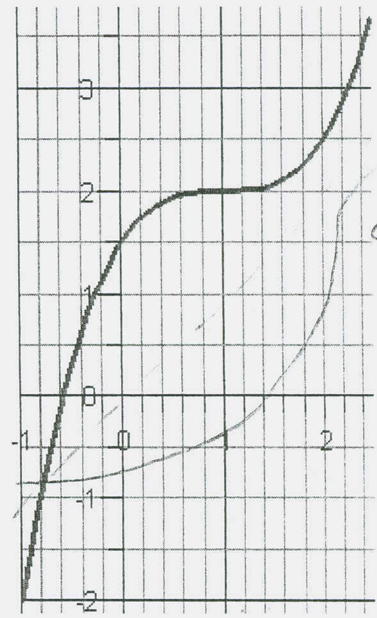
a. (5 pts) If f is one-to-one, with domain **R**, then $f^{-1}(f(6)) = 6$.

True. $f^{-1}(f(x)) = x \quad \forall\, x \in \mathcal{D}(f)$

b. (5 pts) If f is one-to-one and differentiable, with domain **R**, then $\left(f^{-1}\right)'(6) = \dfrac{1}{f'(6)}$.

False. $(f^{-1})'(6) = \dfrac{1}{f'(f^{-1}(6))}$

2. The graph of g is given.



a. (5 pts) Why is g one-to-one?

It's increasing.

b. (5 pts) Estimate the value of $g^{-1}(1)$.

$g^{-1}(1) \approx -.2$

c. (5 pts) Sketch the graph of g^{-1}.

sorta.

3. Find the exact value of each of the following:

a. (5 pts) $\ln\left(e^{\pi}\right) = \pi$

b. (5 pts) $\cos\left(\arctan\sqrt{3}\right) = \frac{1}{2}$

4. Solve the following equations for x.

a. (5 pts) $\ln\left(1 + e^{-x}\right) = 3$

$1 + e^{-x} = e^{3}$

$e^{-x} = e^{3} - 1$

$-x = \ln(e^{3} - 1)$

$x = -\ln(e^{3} - 1)$

b. (5 pts) $\ln(x+1)+\ln(x-1)=1$

$\ln(x^2-1) = 1$

$x^2-1 = e$

$x^2 = e+1$

$x = \pm\sqrt{e+1}$

But $x>1$, by domain of $\ln(x-1)$, so $x=-\sqrt{e+1}$ $\nexists$. Hence $x=\sqrt{e+1}$

5. Differentiate. *Do Not Simplify.*

a. (4 pts) $f(t) = t^2 \ln t$

$f'(t) = 2t\ln t + t^2 \cdot \frac{1}{t}$

b. (4 pts) $g(x) = 3^{mx}\cos(nx)$

$g'(x) = (\ln 3)(m)(3^{mx})\cos(nx) - (3^{mx})(\sin(nx))(n)$

c. (4 pts) $V(t) = \arctan\left(\arcsin\sqrt{t}\right)$

$V'(t) = \frac{1}{1+(\arcsin\sqrt{t})^2} \cdot \frac{1}{\sqrt{1-(\sqrt{t})^2}} \cdot \frac{1}{2\sqrt{t}}$

d. (4 pts) $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$ (Use logarithmic differentiation.)

$\ln y = 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1)$

$\frac{y'}{y} = 4\left(\frac{2x}{x^2+1}\right) - 3\left(\frac{2}{2x+1}\right) - 5\cdot\frac{3}{3x-1}$

$\rightarrow y' = \left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)\left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right)$

6. Cobalt-60 has a half-life of 5.24 years.

a. (4 pts) Find the mass that remains from a 100-mg sample after t years. ~~Your answer should be to the nearest 10th of a gram.~~

$A_0 e^{5.24k} = \frac{1}{2}A_0$

$\boxed{A(t) = 100e^{kt}}$

$e^{5.24k} = \frac{1}{2}$

$5.24k = \ln(\frac{1}{2})$

$\boxed{k = \frac{\ln(\frac{1}{2})}{5.24}} \approx \boxed{-.1322799963 \approx k}$

b. (3 pts) How long would it take for the mass to decay to 1 mg? Your answer should be to the nearest year.

$A(t) = 1$

$100e^{kt} = 1$

$e^{kt} = \frac{1}{100}$

$kt = \ln(\frac{1}{100})$

$t = \frac{1}{k}\ln(\frac{1}{100})$

$= \frac{5.24}{\ln(\frac{1}{2})}\ln(\frac{1}{100}) \approx 34.8$ yrs

$\boxed{35 \text{ yrs}}$

c. (3 pts) What is the rate of decay after 10 years? What is the *percentage* rate of decay?

$A'(10) = ?$

$A'(t) = 100ke^{kt}$

% rate is $\boxed{-13.22799963\ \%}$

$A'(100) = 100ke^{100k}$

$= (100)\frac{\ln(\frac{1}{2})}{5.24}e^{(10)\frac{\ln(\frac{1}{2})}{5.24}} \approx \boxed{-3.523785301\ g/yr}$

7. (4 pts) Find $f'(x)$ for $f(x) = \int_1^{\sqrt{x}} \frac{e^t}{t}dt$

$f'(x) = \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

8. Evaluate the integral.

a. (5 pts) $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{du}{1+u^2} = \arctan u \Big]_0^1$

$u = \sin x$

$du = \cos x\, dx$

$\sin(0) = 0$

$\sin(\frac{\pi}{2}) = 1$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

b. (5 pts) $\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\arcsin(x^2) + C$

c. (5 pts) $\int \ln(\cos x)\tan x dx = -\int u\,du = -\frac{u^2}{2} + C$

$u = \ln(\cos x)$

$du = \frac{-\sin x}{\cos x}\,dx$

$= -\tan x\,dx$

$= -\frac{(\ln(\cos x))^2}{2} + C$

9. (5 pts) If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

$f(x) \overset{\text{SET}}{=} 1$

$\Rightarrow x = 0$

since $0 + 0^2 + e^0 = 1$

So, $f^{-1}(1) = 0$

$f'(x) = 1 + 2x + e^x$

$f'(0) = 1 + e^0 = 2$

$\Rightarrow (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{2}$

10. (5 pts) If $\tanh x = \frac{3}{5}$, find the value of the other 5 hyperbolic trigonometric functions. This should not require a calculator.

$$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{9}{25} = \frac{25-9}{25} = \frac{16}{25}$$

$\Rightarrow \operatorname{sech} x = \pm \frac{4}{5}$

$\Rightarrow \boxed{\operatorname{sech} x = \frac{4}{5}}$

$\Rightarrow \boxed{\cosh x = \frac{5}{4}}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{4}{5} \sinh x = \frac{3}{5}$

$\Rightarrow \boxed{\sinh x = \frac{3}{4}}$

$\Rightarrow \boxed{\operatorname{csch} x = \frac{4}{3}}$

& $\boxed{\coth x = \frac{5}{3}}$

11. (Bonus) (5 pts) Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x+n)e^x$.

$\boxed{\text{Pf}}$ Let $f(x) = xe^x$. Then $f'(x) = f^{(1)}(x) = e^x + xe^x$ $= (1+x)e^x$. & $f^{(k)}(x) = (k+x)e^x = (x+k)e^x$.

$$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \frac{d}{dx}\left[(k+x)e^x\right] = e^x + (k+x)e^x$$

$$= (1+k+x)e^x = ((k+1)+x)e^x = (x+(k+1))e^x \quad \blacksquare$$