where we left off:

#31 in text

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{((n+1)!)^k}{(k(n+1))!} \cdot \frac{(kn)!}{(n!)^k} |x|$$

$$= \left(\frac{(n+1)!}{n!}\right)^k \left(\frac{(kn)!}{\underbrace{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}_{k \text{ factors}} \cdot (kn)!} \right) |x|$$

$$= (n+1)^k \frac{1}{(kn+k)(kn+k-1)(kn+k-2)\cdots(kn+2)(kn+1)} |x|$$

$$= \frac{(n+1)^k}{n(k+\frac{k}{n})n(k+\frac{k-1}{n})n(k+\frac{k-2}{n})\cdots n(k+\frac{2}{n})n(k+\frac{1}{n})} |x|$$

$$= \frac{(n+1)^k}{n^k} \frac{1}{(k+\frac{k}{n})(k+\frac{k-1}{n})\cdots(k+\frac{2}{n})(k+\frac{1}{n})} |x|$$

$$\xrightarrow{n \to \infty} \frac{1}{k^k}|x| \overset{\text{want}}{<} 1 \implies$$

$$|x| < \boxed{k^k = R}$$

S 11.9 - Representation of Functions as Power Series, or...

Everything you wanted to know about $f(x) = \frac{1}{1-x}$

Recall geometric sum:

$$\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right). \quad \text{Let } x = r, \ |x| < 1 \Rightarrow$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Very Cool.

So, $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$!

$$\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n (2)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}$$

Term-by-term differentiation and integration.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

To be determined by evaluating @ $x=a$.

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 5

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right] = -1\,(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}, \text{ so}$$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$ & we have the power series for $\frac{1}{1-x}$!

$$\frac{d}{dx}\left[1 + x + x^2 + x^3 + \cdots\right] = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)x^n = \boxed{\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}}$$

Example 6

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C$$

What's power series for $\ln(1+x)$?

$$\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Thanks to Jason..

$$\Rightarrow \ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

& $\ln(1+0) = 0 = C$

So $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$\underbrace{(-1)^{n-1},\ (-1)^{n+1}}$

Same purpose. even terms negative.

$\boxed{E7}$ Power Series for $\arctan(x)$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

So $\arctan(x) = \sum_{n=0}^{\infty} \int ((-1)^n x^{2n})\,dx = \int \frac{dx}{1+x^2}$

$$\frac{1}{9+x} = \frac{1}{9(1+\frac{x}{9})} = \frac{1}{9}\cdot\frac{1}{1-(-\frac{x}{9})}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{x}{9}\right)^n = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{9}\right)^n x^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{9}\right)^{n+1}x^n$$

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

29. 0/1 points SCalc9 11.9.518.XP. [4592

Use the formula to compute ln(1.08) correct to five decimal places.

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

ln(1.08) = ✗ 0.07696

$$\ln(1+x) = \ln(1-(-x)) = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n} = -\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}$$

$$\ln(1.08) = \ln(1+.08) = -\sum_{n=1}^{\infty}(-1)^n\frac{(.08)^n}{n}$$

Correct to 5 places means error < 5 in the 6th digit.

$$\text{Error} < .000005 = \frac{5}{10^6} = 5\times10^{-6}$$

$R_n \le b_{n+1}$ for alternating series

$$\frac{(.08)^n}{n} < 5\times10^{-6}$$ missed

$$\frac{(8\times10^{-2})^n\times10^6}{5} = \frac{(.08)^n}{5\times10^{-6}} < n$$

$$= \frac{8^n\times10^{-2n}\times10^6}{5} = \cancel{8\times10^3\times2 = 16\times10^3 = 16{,}000}$$

Seems like a huge number. → It is.

Start Over

$$\frac{(.08)^n}{n} < .000005$$

$$\boxed{.08^n < .000005\,n} \rightarrow$$

$$\left(\frac{2}{25}\right)^n = \left(\frac{8}{100}\right)^n < 5\times10^{-6}\,n$$

$$\frac{8^n}{100^n} = \frac{8^n}{10^{2n}} < 5\times10^{-6}\,n$$

$$\left(\frac{2}{25}\right)^n < 5\times10^{-6}\,n$$

$$n\ln\left(\frac{2}{25}\right) < \ln(5) + \ln(10^{-6}) + \ln(n)$$

$solve(0.08^x - 0.000005\,x = 0)$ 4.258985495

Technology says $n+1 \approx 4.26$, so $n+1=5$ will do it.

THAT MEANS $\boxed{n=4}$

In actual practice, without having to solve, just crank out terms until you get one that's smaller than the error.

Error of an alternating series: $\sum_{k=1}^{n}(-1)^k b_k = S_n$

$$\text{\&}\ R_n = |S' - S_n| < b_{n+1}$$

#27

$$\int \frac{x - \arctan(x)}{x}\, dx$$

arctan(x) power series

$$\frac{1}{1-(-x^2)} = \frac{d}{dx}\left[\arctan(x)\right]$$

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{d}{dx}\left[\arctan(x)\right]$$

$$\cancel{\Rightarrow \arctan(x) = \int \left[1 - x + x^2 - x^3 + \dots\right] dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}$$

$$\cancel{= \sum_{n=1}^{\infty} (-1)^{n-1} x^n = \arctan}$$

No. $\sum \int (-1)^n x^{2n}\, dx$

$$\int \left[1 - x^2 + x^4 - x^6 + x^8 - \dots\right] dx = \arctan(x)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$2n+1$: 1, 3, 5 ✓

So $$\frac{x - \arctan(x)}{x} = 1 - \frac{1}{x}\arctan(x)$$

$$= 1 - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$$

So $$\int \frac{x - \arctan(x)}{x}\, dx = x - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$$

Almost there. Need to incorporate the "x" into the sum.

$$x - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} = x + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)^2}$$

$$x = (-1)^0 \frac{x^{2(0)+1}}{(2(0)+1)^2}$$

$$= x - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} \dots$$

49, 25, 9 Squared!

$$= \sum_{n=1}^{\infty} \left(\frac{x^{2n+1}}{(2n+1)^2}\right)(-1)^{n+1}$$

Getting index right is drain bramaging.