S 11.8 - Power Series

$$c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

x is just a variable, here.

$S_n = \sum_{k=0}^{n} c_k x^k$ is n^{th} partial sum

$= n^{\text{th}}$-degree polynomial!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad !$$

More generally, $\sum_{n=0}^{\infty} c_n (x-a)^n$

Power series centered @ $x = a$.

All power series centered @ $x = a$ converge @ $x = a$, trivially:

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \Rightarrow f(a) = c_0$!, using the convention that $(x-a)^0 = 1$.

$\left.\begin{array}{l}\text{Radius of Convergence} \\ \text{Interval of convergence}\end{array}\right\}$ about $x = a$.

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$ **4.** $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

(3) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n(nx^n)}\right| = \frac{n+1}{n}|x| \xrightarrow{n\to\infty} |x| \overset{\text{WANT}}{<} 1$

$|x| < 1$ means $1 + \frac{1}{n} \xrightarrow{n\to\infty} 1$

$x < 1$ and $x > -1$ $\left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e$

& writing $-1 < x < 1$ is OK

I tend to discourage this $-1 < x < 1$, b/c

$|x| > 1$ means

$x > 1$ OR $x < -1$

& writing $-1 > x > 1$ is BAD. It implies $-1 > 1$

$|x| < 1 \rightarrow$
$-1 < x < 1$. So $\boxed{R = 1}$ & "middle" $= a = 0$.

We don't know if it converges @ $x = \pm 1$ without testing.

$f(x) = \sum_{n=0}^{\infty} (-1)^n n x^n$

$f(-1) = \sum_{n=0}^{\infty} (-1)^n n (-1)^n = \sum_{n=0}^{\infty} n \not\rightarrow$

$f(1) = \sum_{n=0}^{\infty} (-1)^n n (1)^n = \sum_{n=0}^{\infty} (-1)^n n$ Fails Test For Divergence.

Interval of Convergence $\boxed{I = (-1, 1)}$

(4) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}} = f(x)$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\right)\left(\frac{\sqrt[3]{n}}{(-1)^n x^n}\right)\right| = \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}}|x| \xrightarrow{n\to\infty} |x| \overset{\text{want}}{<} 1$

$\boxed{R = 1}$

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt[3]{n}} \longrightarrow \text{by Alt. Series Test.}$$

$1 \in I$.

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} \not\longrightarrow \text{by p-Test.}$$

$p = \frac{1}{3}$ - Test

So $\boxed{I = (-1, 1]}$

$$\sqrt[3]{\frac{n}{n+1}} \xrightarrow{n \to \infty} 1$$

$$\lim \sqrt[3]{\frac{n}{n+1}}$$

$$= \sqrt[3]{\lim \frac{n}{n+1}}$$

c is fixed $\Longrightarrow$ $\lim (a_n^c) = (\lim a_n)^c$

$\lim (a_n^{\frac{1}{c}}) = (\lim a_n)^{\frac{1}{c}}$

$\boxed{9.\ \sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n} = f(x)}$

10. $\sum_{n=1}^{\infty} 2^n n^2 x^n$

$2\left(\frac{n+1}{n}\right)^2 |x| \xrightarrow{n\to\infty}$

$2|x| < 1 \Rightarrow \boxed{R = \frac{1}{2}}$

9 $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^4 4^{n+1}} \cdot \frac{n^4 4^n}{x^n}\right| = \left|\frac{n^4}{(n+1)^4} \frac{4^n}{4^n 4^1} \frac{x^n x^1}{x^n}\right|$

$= \left(\frac{n}{n+1}\right)^4 \left(\frac{1}{4}\right) |x| \xrightarrow{n\to\infty} \frac{1}{4}|x| < 1 \Rightarrow |x| < \boxed{4 = R}$

$(ab)^c = a^c b^c$

$-4 < x < 4$

$f(-4) = \sum_{n=1}^{\infty} \frac{(-4)^n}{n^4 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (4)^n}{n^4 (4)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \longrightarrow$ By Alt. Series Test

so $-4 \in I$ & $f(4) = \dots \sum_{n=1}^{\infty} \frac{1}{n^4} \longrightarrow$ by $4 =$ p-test

& hence $\boxed{I = [-4, 4]}$

10 $\sum 2^n n^2 x^n$ $R = \frac{1}{2}, a = 0$

$-\frac{1}{2} < x < \frac{1}{2}$

~~Sloppy &~~ ~~Cryptic.~~

$f(-\frac{1}{2}) = \sum 2^n n^2 (-2)^{-n} = \sum n^2 (-1)^n \not\to$

$f(\frac{1}{2}) = \sum 2^n n^2 (\frac{1}{2})^n = \sum n^2 \left(\frac{2^n}{2^n}\right) = \sum n^2 \not\to$

So $\boxed{I = (-\frac{1}{2}, \frac{1}{2})}$

If you're GOOD, you can be explicit.

31. If k is a positive integer, find the radius of convergence of the series

$$f(x) = \sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n \qquad |x|(n+1)^K$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)!)^K}{(n!)^K} \frac{(K(n+1))!}{(Kn)!} \frac{x^{n+1}}{x^n}\right| = \left(\frac{(n+1)!}{n!}\right)^K \cdots \text{ Go To } (*)$$

$$(K(n+1))! = (Kn+K)!$$

$$\frac{(K(n+1))!}{(Kn)!} = \frac{(Kn+K)(Kn+(K-1))(Kn+(K-2))\cdots(Kn+(K-(K-1)))\cancel{(Kn)!}}{\cancel{(Kn)!}}$$

$$> \underbrace{(Kn)(Kn)(Kn)\cdots(Kn)}_{K \text{ of 'em}} = (Kn)^K = K^K n^K \quad \text{No help!}$$

Try:

$$= K(n+1)\ldots \text{ No Help} \ldots$$

$$= (Kn+K)(Kn+K-1)(Kn+K-2)\cdots(Kn+2)(Kn+1)$$

$$< (Kn+K)(Kn+K)(Kn+K)\cdots(Kn+K)$$

$$= \underbrace{K(n+1)K(n+1)K(n+1)\cdots K(n+1)}_{K \text{ of 'em}} = K^K n^K$$

(*)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)!)^K}{(n!)^K} \frac{(K(n+1))!}{(Kn)!} \frac{x^{n+1}}{x^n}\right| < \left(\frac{(n+1)!}{n!}\right)^K K^K n^K |x| < 1$$

Fly in ointment.

$K = 5$

$$\frac{(5(n+1))!}{(5n)!} = \frac{(5(n+1))(5(n+1)-1)(5n+3)(5n+2)(5n+1)\cancel{(5n)!}}{\cancel{(5n)(5n-1)(5n-2)\cdots(3)(2)(1)}} \qquad 5n+5-1$$

Want

$$\frac{|x|}{K^K} < 1, \text{ given } R = K^K \text{ from sol'ns.}$$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

$$\frac{((n+1)!)^K}{(K(n+1))!} \cdot \frac{(Kn)!}{(n!)^K} |x|$$

$$= \left(\frac{(n+1)!}{n!}\right)^K \cdot \frac{\cancel{(Kn)(Kn-1)\cdots(2)(1)}\;|x|}{(Kn+K)(Kn+K-1)\cdots(Kn+2)(Kn+1)\cancel{(Kn)(Kn-1)}\cdots}$$

$$Kn+K-2$$
$$K(n+1)-\frac{2}{K}$$

$$= (n+1)^K \frac{|x|}{K^K(n+1)(n+1-\frac{1}{K})(n+1-\frac{2}{K})\cdots}$$

Ryan Cheats and looks it up, here:

https://math.stackexchange.com/questions/888163/finding-the-convergence-radius-of-dfracnk-cdot-xnkn

Factor n out of each factor in the denominator in the ratio test and it falls into our laps. Nice work, people!

Just looking at the answer in the back of my book got me no closer to a cogent argument.

2 wrongs are just 2 wrongs.

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

S 11.9 Part I - Everything you wanted to know about $\frac{1}{1-x}$