1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\Sigma\,(-1)^{n-1}b_n$ or $\Sigma\,(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

most frequently used test for power series in the sequel.

$$\sum_{k=1}^{n} \frac{x^{k-1}}{k}$$

For what x does this converge?

$$\left|\left(\frac{x^{k+1}}{k+1}\right)\left(\frac{k}{x^k}\right)\right| = \left|\frac{k}{k+1} \times\right| \xrightarrow{k\to\infty} |x| \underset{<}{\text{NEED}}\ 1$$

we need $|x| < 1$

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

9. If Ratio Test is inconclusive, so will the Root Test be inconclusive.

Love,

Ryan

$\int \tan(u)\,du = \ln|\sec(u)| + C$

#23 in the text:

$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

$\tan\left(\frac{1}{n}\right) \xrightarrow{n\to\infty} 0$ See video

Text compares it to $\sum_{n=1}^{\infty} \frac{1}{n}$ in Limit Comparison.

I'm not sure where you find the inspiration to compare to 1/n when looking at tan(1/n).

$$\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{n\to\infty} \frac{0}{0} \xrightarrow[L'H]{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \sec^2\left(\frac{1}{n}\right) \xrightarrow{n\to\infty} 1$$

Easy-Peasie!

Insight: TRY limit comparison on one or two that you know. When in doubt, try a p-series.

p=1 on 1st try is pretty clever.

p=2:

$$\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \xrightarrow[L'H]{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{-2n^{-3}} = \frac{\sec^2\left(\frac{1}{n}\right)}{\frac{2}{n}} = \frac{n}{2}\sec^2\left(\frac{1}{n}\right)$$

$$-2n^{-3} = \frac{-2}{n^3}$$

$\xrightarrow{n\to\infty} \infty$, so

$\tan\left(\frac{1}{n}\right)$ way bigger than $\frac{1}{n^2}$, eventually.

$$\frac{-\frac{1}{n^2}}{-\frac{2}{n^3}} = \left(\frac{1}{n^2}\right)\left(\frac{n^3}{2}\right) = \frac{n}{2}$$

$\int \tan\left(\frac{1}{x}\right) dx$

No. You're thinking arctan(u), here, dummy

$u = \tan\left(\frac{1}{x}\right)$

$du = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) dx = \left(\frac{1}{\frac{x^2+1}{x^2}}\right)\left(-\frac{1}{x^2}\right) dx$

$= \left(\frac{x^2}{x^2+1}\right)\left(-\frac{1}{x^2}\,dx\right) =$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Ryan says "~~Integral Test.~~" ↳ Probably an optimal choice.

$u = -n^3$
$du = -3n^2\,dn$ is staring you in the face

$$\left|\frac{(n+1)^2 e^{-(n+1)^3}}{n^2 e^{-n^3}}\right| = \left(\frac{n^2+2n+1}{n^2}\right)\left(\frac{e^{n^3}}{e^{(n+1)^3}}\right)$$

$$= \left(\frac{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{n^2}\right)\left(\frac{\cancel{e^{n^3}}\;1}{(\cancel{e^{n^3}})(e^{3n^2})(e^{3n})e^1}\right) \xrightarrow{n\to\infty} \frac{1}{\infty} \text{ "=" } 0$$

e^{3n^2+3n+1}

n^3+3n^2+3n+1

6. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

Intuition: $\frac{n^{2n}}{(1+n)^{3n}} \approx \frac{n^{2n}}{n^{3n}} = \frac{1}{n^{3n-2n}} = \frac{1}{n^n}$

hella convergent.

$$\frac{n^{2n+2}}{(2+n)^{3n+3}} \cdot \frac{(1+n)^{3n}}{n^{2n}} = \frac{n^2}{\text{ugh.}} \quad \frac{(1+n)^{3n}}{(2+n)^{3n+3}}$$

doesn't cancel, neatly.

6. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} = \sum_{n=1}^{\infty} \frac{(n^2)^n}{((n+1)^3)^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{n^2}{(n+1)^3}\right)^n$$

Root test: $\sqrt[n]{|a_n|} = \frac{n^2}{(n+1)^3} \xrightarrow{n\to\infty} 0$

$$\frac{n^{2n}}{(n+1)^{3n}} < \frac{n^{2n}}{n^{3n}} = \left(\frac{n^2}{n^3}\right)^n = \left(\frac{1}{n}\right)^n$$

Root Test: $\sqrt[n]{|a_n|} = \frac{1}{n} \xrightarrow{n\to\infty} 0 < 1$

$\Longrightarrow$ Convergent.

Exponents – Logarithms

14. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

$\left|\frac{\sin(2n)}{1+2^n}\right| \leq \left|\frac{1}{1+2^n}\right| < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, so

compares favorably to convergent geometric series

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\cdot\frac{1^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$a = \frac{1}{2}$

$r = \frac{1}{2}$

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$

Read about the Cantor Set.

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1\cdot3\cdot5\cdots(2n-1)(2(n+1)-1)}{2\cdot5\cdot8\cdots(3n-1)(3(n+1)-1)} \cdot \frac{2\cdot5\cdot8\cdots(3n-1)}{1\cdot3\cdot5\cdots(2n-1)}$$

$$= \frac{2(n+1)-1}{3(n+1)-1} = \frac{2n+2-1}{3n+3-1} = \frac{2n+1}{3n+2} \xrightarrow{n\to\infty} \frac{2}{3} < 1$$

Converges!

33. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Memorize $\left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e$

$\left(\frac{n+1}{n}\right)^n \xrightarrow{n\to\infty} e$

$$\left(\frac{n}{n+1}\right)^{n^2} = \left(\frac{n+1}{n}\right)^{-n^2} = \left(\left(\frac{n+1}{n}\right)^n\right)^{-n}$$

$$= \left(\left(1+\frac{1}{n}\right)^n\right)^{-n} = \left(\boxed{\left(\left(1+\frac{1}{n}\right)^n\right)^{-1}}\right)^n$$

$\xrightarrow{n\to\infty} 0$ e^{-1} in the limit (A bit sketchy)

33. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\sqrt[n]{\left(\left(\frac{n}{n+1}\right)^n\right)^n} = \left(\frac{n}{n+1}\right)^n = \left(\left(\frac{n+1}{n}\right)^n\right)^{-1}$$

$\xrightarrow{n\to\infty} e^{-1} = \frac{1}{e} < 1$ converges

ROOT TEST WORKED WELL.

34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ $\boxed{D}$

$\frac{1}{n+n\cos^2(n)} > \frac{1}{n+n} = \frac{1}{2n}$ & $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln(n+1)^{\ln(n+1)}}{\ln(n)^{\ln(n)}}$$

$$\int \frac{dn}{\ln(n)^{\ln(n)}}$$

$u = \ln(n)$

$du = \frac{1}{n}dn$

$$\sqrt[n]{(\ln(n))^{\ln(n)}} = \ln(n)^{\frac{\ln(n)}{n}} = y$$

$n^2 < \ln(n)^{\ln(n)}$?

$n^2 - \ln(n)^{\ln(n)} = f(n)$

$\ln(n) > 2$, eventually

Good Bonus?!

$f'(n) = 2n - \ln(n)\,\ln(n)^{\ln(n)-1}\left(\frac{1}{n}\right)$

$= \frac{2n^2 - \ln(n)^{\ln(n)}}{n}$

Note $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n} > n^2$ for n sufficiently large (i.e. $\ln \ln n > 2$ or $n > 1619$). Thus

$$\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$$

and thus the series converges by the comparison test.

I think Victoria was on the right track. We were (I was) just struggling to understand.