1 Definition A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

2 Definition A series Σa_n is called **conditionally convergent** if it is convergent but not absolutely convergent.

Conditionally convergent series:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\ln(k)}$$

2–6 Determine whether the series is absolutely convergent or conditionally convergent.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ cond.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

Jocelyn

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

7–24 Use the Ratio Test to determine whether the series is convergent or divergent.

7. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

8. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

9. $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{3^n}{2^n n^3}$

10. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

11. $\sum_{k=1}^{\infty} \frac{1}{k!}$

12. $\sum_{k=1}^{\infty} ke^{-k}$

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

14. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

15. $\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$

16. $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$

#9 $\frac{3^{n+1}}{2^{n+1}\cdot(n+1)^3}\cdot\frac{2^n\cdot n^3}{3^n}$

$= \frac{3}{2}\cdot\frac{n^3}{(n+1)^3} \xrightarrow{n\to\infty} \frac{3}{2} > 1$

So $\sum |a_n|$ diverges.

what about $\sum a_n$

$\frac{3^n}{2^n n^3} \xrightarrow{n\to\infty} \frac{\infty}{\infty}$

$\left(\frac{3}{2}\right)^n\left(\frac{1}{n^3}\right) = \frac{\left(\frac{3}{2}\right)^n}{n^3}$ is

now ready for L'Hôpital

$$\frac{\left(\frac{3}{2}\right)^n}{n^3} \xrightarrow{n\to\infty} \frac{\infty}{\infty} \xrightarrow[L'H]{n\to\infty} \frac{\ln(3/2)\cdot\left(\frac{3}{2}\right)^n}{3n^2} \xrightarrow[L'H]{n\to\infty}$$

$$\frac{(\ln(3/2))^2\left(\frac{3}{2}\right)^n}{6n} \xrightarrow[L'H]{n\to\infty} \frac{(\ln(3/2))^3\left(\frac{3}{2}\right)^n}{6} \xrightarrow{n\to\infty} \infty$$

ONE way to fairly elegantly execute L'Hôpital's Rule.

$$\left(\frac{3}{2}\right)^n = e^{\ln\left(\left(\frac{3}{2}\right)^n\right)} = e^{(\ln(3/2))n}$$

$$\Rightarrow \frac{d}{dn}\left[\left(\frac{3}{2}\right)^n\right] = \ln(3/2)\cdot e^{(\ln(3/2))n} = \ln\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^n$$

#14

$\sum_{n=1}^{\infty} \frac{n!}{100^n}$. $a_n = \frac{n!}{100^n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}$

$= \frac{(n+1)}{100} \xrightarrow{n\to\infty} \infty$ $\boxed{\text{Diverges}}$

#10 $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $a_n = \frac{n!}{n^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{1} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{n+1}{n+1} \cdot \frac{n^n}{(n+1)^n} = \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n + \text{smaller stuff}}$$

$$n^n + n(n^{n-1}) + \frac{n(n-1)}{2} \cdot n^{n-2} + \ldots$$

$$= \frac{n^n}{n^n\left(1 + \frac{\text{stuff of lower degree}}{n^n}\right)}$$

$\xrightarrow{n\to\infty} 1$ should?

$$\ldots = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \xrightarrow{n\to\infty} \frac{1}{e}\text{, you fool!}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \ldots$$

0 1

$$\binom{n}{2} = \frac{n!}{(2!)(n-2)!} = \frac{n(n-1)}{2}$$

Ryan suggests Example 5 and would like it explained a bit...

Aha! $\frac{n+1}{n} = \frac{n}{n} + \frac{1}{n} = 1 + \frac{1}{n}$ way cool!

Back to # 18:

$$\left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1+\frac{1}{n}\right)^{-n} = \left(\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

$\xrightarrow{n\to\infty} e^{-1} = \frac{1}{e}$, so there's a fallacy in my work on the previous page.

Yes, $\frac{n}{n+1} \xrightarrow{n\to\infty} 1$, but

$\left(\frac{n}{n+1}\right)^n \xrightarrow{n\to\infty} \frac{1}{e}$!

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

25–30 Use the Root Test to determine whether the series is convergent or divergent.

25. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

26. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

27. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

28. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

29. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

30. $\sum_{n=0}^{\infty} (\arctan n)^n$

#26 $\frac{(-1)^n \cdot 2^n}{n^n}$ $(-1)^n \left(\frac{2}{n}\right)^n$

$\sqrt[n]{\left(\frac{2}{n}\right)^n} = \frac{2}{n} \xrightarrow{n\to\infty} 0$

$\boxed{\text{Converges}}$

29 $\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^{n^2\cdot\frac{1}{n}} = \left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e > 1$

Diverges.

Riemann proved that

if Σa_n is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of Σa_n that has a sum equal to r.

See #52 for outline of proof.

Take positive terms, 'til you're above L.
subtract negative terms, 'til you're below L.
Repeat.

Financial Math.

Simple Interest:

$P + Pr = P(1+r) = A =$ Future value

Compound Interest: $i = \frac{r}{m}$, $r =$ APR, $m = \#$ of periods per year.

Period | A
1 | $P(1+i)$
2 | $P(1+i) + P(1+i)i = P(1+i)(1+i) = P(1+i)^2$
$\vdots$
n | $P(1+i)^n$, when $n = \#$ of periods

Annuity: $R =$ monthly (periodic pmt)
$i = \frac{r}{m}$ as before.
$n = \#$ of periods.

Period
1 R
2 $R + R(1+i)$
3 $R + R(1+i) + R(1+i)^2$
4 $R + R(1+i) + R(1+i)^2 + R(1+i)^3$
$\vdots$
n $R + R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-1}$ $= \frac{a(1-r^n)}{1-r}$

After n periods $FV = \sum_{k=1}^{n} R(1+i)^{k-1}$ $= \frac{R(1-(1+i)^n)}{1-(1+i)}$

$a = R$
$r = 1+i$ $= R\left[\frac{(1+i)^n - 1}{i}\right]$

$P =$ Amount Borrowed

Present Value of an annuity

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$ = Present Value of the annuity.

$$\frac{Pi}{1-(1+i)^{-n}} = R$$ Amortization Formula (Payments)

$$Pe^{rt}$$

$$P(1+i)^n = P\left(1+\frac{r}{m}\right)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

$$= P\left(1+\frac{r}{m}\right)^{\left(\frac{m}{r}\right)rt} = P\left(\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right)^{rt} \xrightarrow{m\to\infty} Pe^{rt}$$

$$\left(1+\frac{1}{t}\right)^t \xrightarrow{t\to\infty} e$$

$$(1+t)^{\frac{1}{t}} \xrightarrow{t\to\infty} e$$

Ryan makes good point:
If ratio test is inconclusive, so will the root test & conversely.