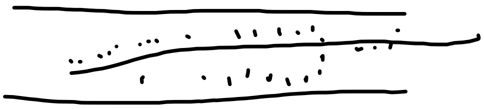
Monotone Convergence: A bounded above (below) increasing (decreasing) sequence converges.

Bounded Convergence: A bounded sequence has a convergent subsequence.

S 11.4



The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

I've already shown you this, out of overeagerness. The Comparison Test is a rather blunt instrument, but you can a lot of insight on "What's bigger?" by using it.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ convergest because $\dfrac{1}{2^n+1} < \dfrac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is trickier, but the same trick I showed you the other day works, here:

$$\frac{1}{2^n - 1} < \frac{1}{2^n - \frac{1}{3}2^n} = \frac{1}{\frac{2}{3}\cdot 2^n} = \frac{3}{2}\cdot\frac{1}{2^n} \;\&$$

$\displaystyle\sum_{n=0}^{\infty} \frac{3}{2}\cdot\frac{1}{2^n}$ is a convergent geometric series

$$= \sum_{n=1}^{\infty}\left(\frac{3}{2}\cdot\frac{1}{2}\right)\left(\frac{1}{2^{n-1}}\right) = \sum_{n=1}^{\infty} \frac{3}{4}\left(\frac{1}{2}\right)^{n-1}, \quad r = \frac{1}{2},\ a = \frac{3}{4}, \text{ etc.}$$

Book will use Limit Comparison to handle the tricky ones

From 11.2:

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Now in the proof of the Comparison Test on Page 768, we have:

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geqslant s_n$). Also $t_n \to t$, so $t_n \leqslant t$ for all n. Since $a_i \leqslant b_i$, we have $s_n \leqslant t_n$.

$>$

Most common comparisons:

to known p-series or known geometric series

The Limit Comparison Test Suppose that Σa_n and Σb_n are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

This gives us a better / more efficient method for $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$. Looks like a $\frac{1}{2^n}$ situation.

$$\frac{\frac{1}{2^{n-1}}}{\frac{1}{2^n}} = \frac{1}{2^{n-1}} \cdot \frac{2^n}{1} = 2^{n-(n-1)} = 2^1 = 2, \text{ so}$$

$$\sum \frac{1}{2^{n-1}} \longrightarrow \text{ iff } \sum \frac{1}{2^n} \longrightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-a} + \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

No!

$a = \frac{1}{2}$, $r = \frac{1}{2}$

$\frac{a}{1-r}$, dummy.

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Comparisons are handy for estimates of sums, as well. Some things are easier to integrate than others.

Estimate $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ with an error no greater than 0.005.

Recall $R_n \leq \int_n^{\infty} f(n)\,dn$

$$a_{n+1} + a_{n+2} + \cdots = R_n = S' - S'_n \leq \int_n^{\infty} f(n)\,dn,$$

where $a_n = f(n)$.

(Monotone decreasing sequence with positive terms.)

Now $T = \sum_{k=1}^{\infty} \frac{1}{2^k}$, $S' = \sum_{k=1}^{\infty} \frac{1}{2^k+1}$

Since $\frac{1}{2^k+1} < \frac{1}{2^k}$ $\forall k \ldots$

$R_n \leq T_n$ → Debating if $<$ is better.

$a_n < b_n$ → $\lim a_n \leq \lim b_n$

$\frac{1}{n^2+1} < \frac{1}{n^2}$, but $\lim \left(\frac{1}{n^2+1}\right) = \lim \left(\frac{1}{n^2}\right) = 0$

$a_{n+1} < b_{n+1}$, $a_{n+2} < b_{n+2}, \ldots$

So, $R_n = \sum_{k=n+1}^{\infty} \frac{1}{2^k+1} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k}$
($<$?)

Want $R_n \leq .005$

$R_n \leq T_n \leq .005$

$T_n \leq \int_n^{\infty} \frac{1}{2^x}\,dx$ is easier to integrate than

$$\int_n^{\infty} \frac{1}{2^x+1}\,dx$$

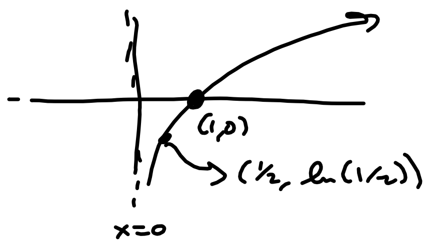
$$\int_n^{\infty} \frac{1}{2^x}\,dx = \int_n^{\infty} \left(\frac{1}{2}\right)^x dx = \int_n^{\infty} e^{\ln\left(\left(\frac{1}{2}\right)^x\right)} dx$$

$$= \int_n^{\infty} e^{x \ln(1/2)}\,dx = \int_n^{\infty} e^{\ln(1/2)\,x}\,dx = \frac{1}{\ln(1/2)} \int_n^{\infty} e^{\ln(1/2)\,x} \cdot \ln(1/2)\,dx$$

$e^{\ln\left(\left(\frac{1}{2}\right)^x\right)} = \left(\frac{1}{2}\right)^x$ $\qquad$ e^u $\qquad$ du

$$= \lim_{t\to\infty} \frac{1}{\ln(1/2)} \left[\left(\tfrac{1}{2}\right)^x\right]_n^t = \frac{1}{\ln(1/2)} \left[\lim_{t\to\infty}\left(\tfrac{1}{2}\right)^t - \left(\tfrac{1}{2}\right)^n\right] < .005$$

$$\Rightarrow \frac{1}{\ln(1/2)}\left[0 - \left(\tfrac{1}{2}\right)^n\right] = -\frac{1}{\ln(1/2)}\left(\tfrac{1}{2}\right)^n = \frac{1}{\ln(2)}\left(\tfrac{1}{2}\right)^n < .005$$



$\ln\left(\frac{1}{2}\right) = \ln(2^{-1})$

$= -\ln(2)$

$$\Rightarrow \quad \left(\tfrac{1}{2}\right)^n < .005 \ln(2)$$

$\sqrt[n]{} = \sqrt[n]{}$ isn't what I want to do

To extract 'n' from exponent, I need to take a log.

$$\ln\left(\left(\tfrac{1}{2}\right)^n\right) = n\ln(1/2) < \ln(.005\ln(2))$$

Yes, Zach. Dividing by a negative reverses the sense of the inequality.

$$n > \frac{\ln(.005\ln(2))}{\ln(1/2)} \approx 8.172622563$$

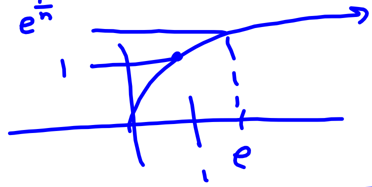
So $n = 9$ will do it.
Maybe $n=8$, since the estimate is rough.

$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ Diverges

$\boxed{e^{1/n} > 1}$ $\forall n$ $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{1}{n} \;\not\to$

$\sqrt[3]{2.71\ldots}$

$e^{\frac{1}{n}}$ $x^{\frac{1}{n}}$



#30 in text

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Good exercise!

$n! = \overbrace{n(n-1)(n-2)\cdots(n-(n-2))(1)}^{n \text{ of 'em}}$

$n^n = \underbrace{n \cdot n \cdot n \cdots n \cdot n}_{n \text{ of 'em}}$

$n^n > n!$, eventually.

$2^2 = 4 > 2 = 2!$

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ We think it diverges.
Let's look at something we know diverges, like $\sum \frac{1}{n}$

$\frac{d}{dn}\left[\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right]$

$-\frac{1}{n^2} - \left(-\frac{1}{n^2}\right)\cos\left(\frac{1}{n}\right) = -\frac{1}{n^2} + \frac{1}{n^2}\cos\left(\frac{1}{n}\right)$

$= \frac{1}{n^2}\left[\cos\left(\frac{1}{n}\right) - 1\right] \le 0$

$\sin\left(\frac{1}{2}\right) > \frac{1}{2}$

& $\frac{d}{dx}\left[\sin\left(\frac{1}{x}\right) - \frac{1}{x}\right] = \frac{1}{n^2}\left[1 - \cos\left(\frac{1}{x}\right)\right] \ge 0$

So $\sin\left(\frac{1}{n}\right) > \frac{1}{n}$ &

$\frac{d}{dx}\left[\sin\left(\frac{1}{x}\right) - \frac{1}{x}\right] \ge 0$ $\therefore$ $\sin\left(\frac{1}{n}\right) > \frac{1}{n} \;\forall n \ge 2$

What I did:

Compared $\sin(\frac{1}{n})$ to $\frac{1}{n}$

Found $\sin(\frac{1}{2}) > \frac{1}{2}$

Proved $\sin(\frac{1}{x}) - \frac{1}{x}$ has positive slope.

This means $\sin(\frac{1}{x})$ <u>STAYS</u> $> \frac{1}{x}$!

$\therefore \sin(1/x) - \frac{1}{x}$ grows as x grows

$\sin(\frac{1}{2}) > \frac{1}{2}$

& $\sin(\frac{1}{x}) - \frac{1}{x}$ has nonnegative slope.

i.e., $\sin(\frac{1}{x})$ grows at least as fast as $\frac{1}{x}$

See "Race track Principle"

See "10-yr cycle on pedagogy"

Will look into Comprehensive Final. Thought I had one up, already...