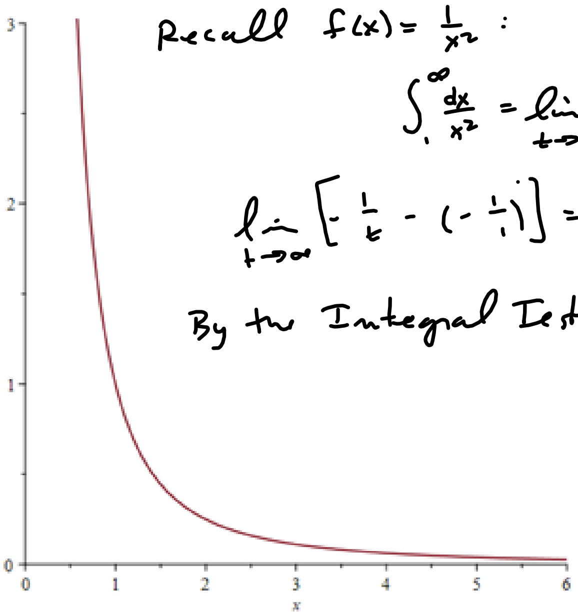
Section 11.3



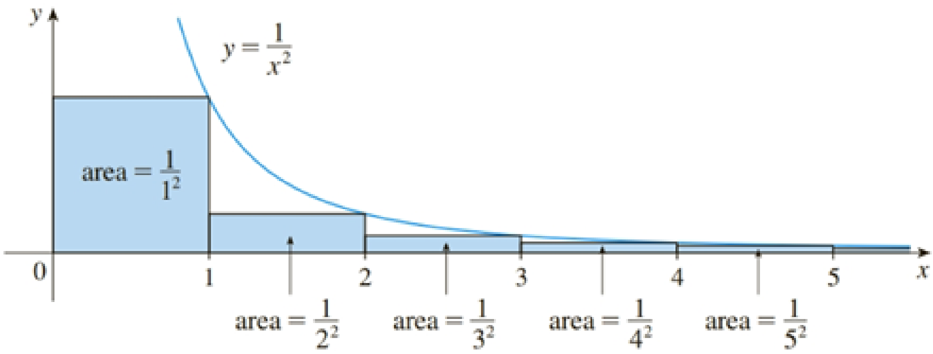
Recall $f(x) = \frac{1}{x^2}$:

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} -\frac{1}{x}\Big]_1^t =$$

$$\lim_{t\to\infty}\left[-\frac{1}{t} - \left(-\frac{1}{1}\right)\right] = 1 \text{ converges } (p\text{-test}, p=2)$$

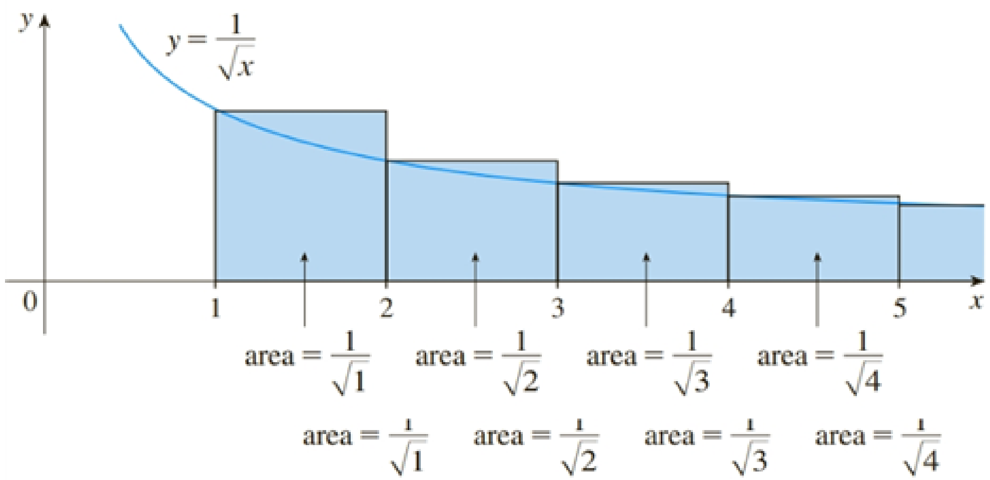
By the Integral Test, then, $\sum_{k=1}^\infty \frac{1}{k^2}$ converges.

$\sum_{k=1}^\infty f(k)$



Note $\sum_{k=1}^\infty \frac{1}{k^2} = 1 + \sum_{k=2}^\infty \frac{1}{k^2} < 1 + \int_1^\infty \frac{dx}{x^2} = 2$

In similar fashion $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges



then, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \int_1^n \frac{dx}{\sqrt{x}}$ & $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges, so $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

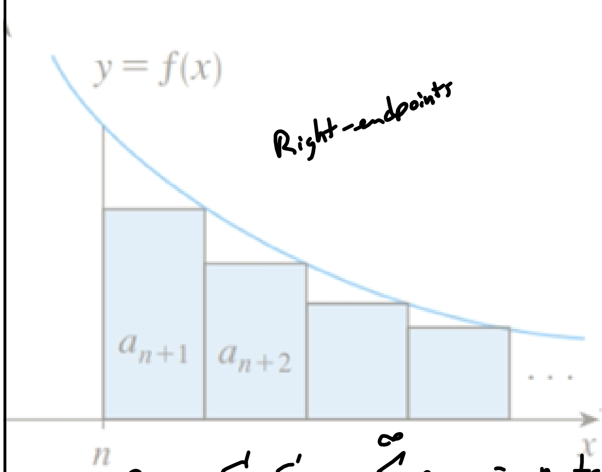
$$\underbrace{\left(\int_1^{\infty} f(x)\,dx \rightarrow\right) \overset{\text{iff}}{\longleftrightarrow} \left(\sum_{k=1}^{\infty} f(k) \rightarrow\right)}_{\text{logically equivalent}}$$

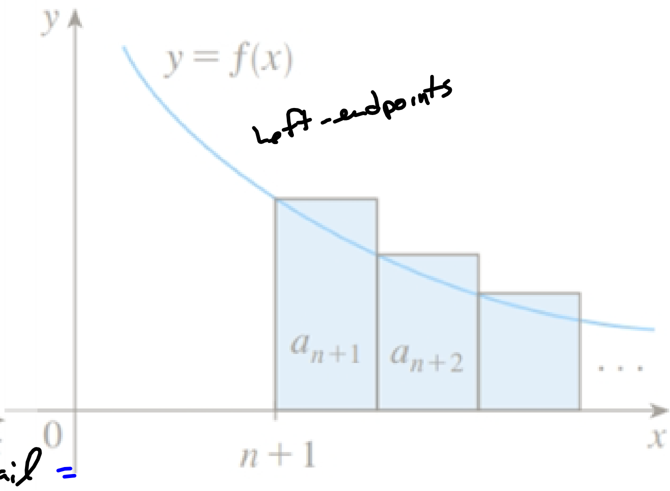
$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

$$\int_1^{\infty} \frac{dx}{x^p} \rightarrow \text{ if } p > 1 \text{ \& } \not\rightarrow \text{ if } p \leq 1$$

p-test

Estimates





$R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k$ = n-tail = $a_{n+1} + a_{n+2} + a_{n+3} + \cdots$ = n-tail = R_n

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

This gives us a way to estimate sums.

11.3 #20

How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^4}$ would you need to add to find its sum to within 0.01?

Accurate to 2 places.
Error less than .005

$\sum_{n=2}$

$R_n < \int_n^{\infty} f(x)\,dx$

$\int_n^t (\ln(x))^{-4}\left(\frac{1}{x}dx\right) = \frac{1}{-3}\ln(x)^{-3}\Big]_n^t = -\frac{1}{3}(\ln(t))^{-3} - (-\frac{1}{3}(\ln(n))^{-3})$

$t \to \infty$ $\frac{1}{3}\ln(n)^{-3} \overset{\text{want}}{<} .005$

$\ln(n)^{-3} < 3(.005) = .015$

$\frac{1}{.015} < \ln(n)^3$

$\sqrt[3]{1/.015} < \ln(n)$

57.67370382 $\approx e^{\sqrt[3]{1/.015}} < n$

$\boxed{\text{So let } n = 58}$

Misinterpreting:
I took it to mean
" we want the 2nd digit to be accurate."
THEY meant " make the error less than or equal to 0.01

So, want $\int_n^{\infty} \ln(x)^{-4}\left(\frac{1}{x}dx\right) = \dots = \frac{1}{3}\ln(n)^{-3} < 0.01$

$\Rightarrow \ln(n)^{-3} < .03 = \frac{3}{100}$

$\frac{100}{3} < \ln(n)^3$

$\sqrt[3]{\frac{100}{3}} < \ln(n)$

$e^{\sqrt[3]{\frac{100}{3}}} < n$

Estimate $\sum_{n=1}^{\infty} (2n+1)^{-5}$ $\boxed{\text{correct to five decimal places.}}$

$\to$ This means $R_n < .000005$

Want $\int_n^{\infty} \frac{1}{(2x+1)^5}\,dx < .000005$

$$\lim_{t\to\infty} \frac{1}{2}\int_n^{\infty} (2x+1)^{-5}(2\,dx) = \lim_{t\to\infty} \left. \frac{\frac{1}{2}\left[2x+1\right]^{-4}}{-4} \right]_n^t$$

$$= -\frac{1}{8}\left[\lim_{t\to\infty} (2t+1)^{-4} - (2n+1)^{-4}\right] = +\frac{1}{8}(2n+1)^{-4} < 0.000005$$

$$\Rightarrow \frac{1}{(2n+1)^4} < 8(5\times10^{-6}) \Rightarrow$$

$$(2n+1)^4 > \frac{1}{8(5)(10^{-6})} = \frac{10^6}{40} = \frac{10^5}{4} \Rightarrow$$

$$2n+1 > \sqrt[4]{\frac{10^5}{4}} = 10\sqrt[4]{\frac{10}{4}} = 10\sqrt[4]{\frac{5}{2}}$$

$$\Rightarrow 2n > 10\sqrt[4]{\tfrac{5}{2}} - 1$$

$$\Rightarrow n > \frac{1}{2}\left[10\sqrt[4]{\tfrac{5}{2}} - 1\right] \approx 5.787167150, \text{ so } n = 6 \text{ terms;}$$

$$\sum_{k=1}^{6}\left(\frac{1}{2k+1}\right)^5$$

$$\frac{8383502097252923134\,24}{185452612752454075153125}$$

$evalf(\%)$

0.004520562947

$$\sum_{n=1}^{\infty} n^4(n^5+1)^p$$ For what p does this converge?

$$\int_1^\infty x^4(x^5+1)^p\,dx = \frac{1}{5}\int_1^\infty (x^5+1)^p(5x^4dx)$$ $u=x^5+1$, $du=5x^4dx$

$$= \frac{1}{5}\int_{1=x}^{\infty} u^p\,du$$ & we need $p<-1$ for this to converge.

Let $q=-p$. Then

$$= \frac{1}{5}\int_{x=1} \frac{du}{u^q}$$ Need $q>1$, i.e., $-p>1$, i.e. $\boxed{p<-1}$

Consider the following function.

$$f(x) = \frac{4\cos(\pi x)}{\sqrt{x}}$$ Corresponds to $\sum_{k=1}^{\infty} \frac{4\cos(\pi k)}{\sqrt{k}}$

$$= \frac{4\cos\pi}{\sqrt{1}} + \frac{4\cos(2\pi)}{\sqrt{2}} + \frac{4\cos(3\pi)}{\sqrt{3}} + \cdots$$

$$= -4 + \frac{4}{\sqrt{2}} - \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{4}} - \cdots + (-1)^k\frac{4}{\sqrt{k}} + \cdots$$

Relative to §11.3, The Integral Test does not apply.

Not decreasing. Not all positive terms.

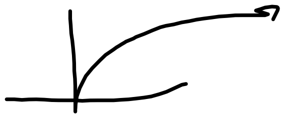
But it DOES converge!

See Alternating Series!

$$\sum_{n=2}^{\infty} \frac{4}{n\ln(n)} \qquad 4\int_2^{\infty} \ln(n)^{-1}\left(\frac{1}{n}dn\right)$$

$$\lim_{t\to\infty} 4\ln(\ln(n))\Big]_2^t = \lim_{t\to\infty}\left[4\ln(\ln(t)) - 4\ln(\ln(2))\right] \quad \nexists$$

$n^{.0000001} > \ln(n)$, eventually.



$n^p > \ln(n)$ eventually, $\forall\, p > 0$.

§11.4

The Comparison Test Suppose that Σa_n and Σb_n are series with positive terms.

(i) If Σb_n is convergent and $a_n \leq b_n$ for all n, then Σa_n is also convergent.

(ii) If Σb_n is divergent and $a_n \geq b_n$ for all n, then Σa_n is also divergent.

$\sum_{k=1}^{\infty} \frac{k^2}{k^5+k^3+k^2}$ converges, by comparison

$a_k = \frac{k^2}{k^5+k^3+k^2} < \frac{k^2}{k^5} = \frac{1}{k^3}$ $\quad \sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, by p=3-test.

$\sum \frac{k^3}{3k^4-5k^3-k^2}$ diverges, by comparison to $\frac{1}{3k}$

$a_k = \frac{k^3}{3k^4-5k^3-k^2} > \frac{k^3}{3k^4} = \frac{1}{3k}$ & $\sum_{k=1}^{\infty} \frac{1}{3k} \not\to$, by p=1-test.

$\sum_{k=1}^{\infty} \frac{k^3}{k^5-3k^2}$ is hard to do by direct comparison.

Intuition: $\sum \frac{k^3 + \sim}{k^5 + \sim}$ behaves like $\sum \frac{1}{k^2}$

Showing $\frac{k^3}{k^5-3k^2}$ < something that converges,

is hard, because throwing away $-3k^2$ makes the fraction bigger, i.e.,

$$\frac{k^3}{k^5-3k^2} > \frac{k^3}{k^5} = \frac{1}{k^2} \text{ is no help}$$

& that's where I threw this trick at you

$$\frac{k^3}{k^5-3k^2} < \frac{k^3}{k^5-3\left(\frac{k^5}{6}\right)} = \frac{k^3}{k^5-\frac{1}{2}k^5} = \frac{k^3}{\frac{1}{2}k^5} = \frac{2}{k^2} \quad \& \quad \sum \frac{2}{k^2} \longrightarrow.$$

The trick. Subtract something bigger, downstairs.

Likewise $\sum \frac{k^3}{k^4+2k^3+k}$

$\frac{k^3}{k^4+2k^3+k}$ > something that DIVERGES.

$\frac{k^3}{k^4+2k^3+k} < \frac{k^3}{k^4}$ Want something smaller that still blows up.

$\frac{k^3}{k^4+2k^3+k}$ WANT $>$ something that diverges.

$$\frac{k^3}{k^4+2k^3+k} > \frac{k^3}{k^4+2k^4+k^4} = \frac{k^3}{4k^4} = \frac{1}{4k} \quad \& \quad \frac{1}{4}\sum \frac{1}{k} \not\longleftrightarrow$$

Bigger denominator makes smaller fraction.