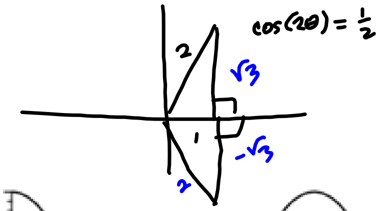
CAUTION The fact that a single point has many representations in par coordinates sometimes makes it difficult to find all the points of intersection of tw polar curves. For instance, it is obvious from Figure 5 that the circle and the cardiid have three points of intersection; however, in Example 2 we solved the equations $= 3\sin\theta$ and $r = 1 + \sin\theta$ and found only two such points, $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. T: origin is also a point of intersection, but we can't find it by solving the equations of theurves because the origin has no single representation in polar coordinates that satisfies oth equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = \sin\theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + si\theta$ and so it lies on the cardioid. Think of two points moving along the curves as the paimeter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origi because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recomrended that you draw the graphs of both curves. It is especially convenient to use a graphig calculator or computer to help with this task.

Example 3 illustrates this point

$r = \cos(2\theta)$ & $r = \frac{1}{2}$

want all $\theta \in [0, 2\pi) \ni \cos(2\theta) = \frac{1}{2}$

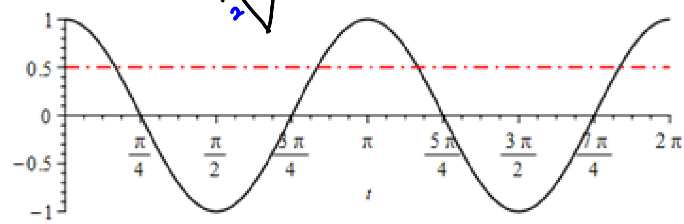


$0 \le \theta < 2\pi$

$0 \le 2\theta < 4\pi$

$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

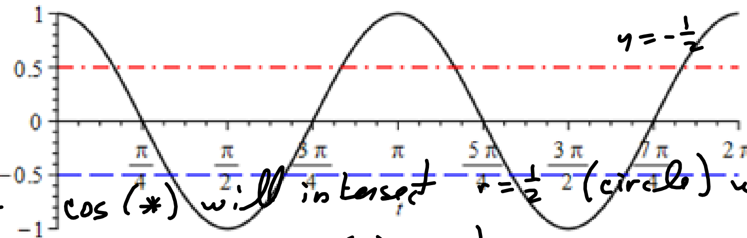
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$



$\frac{\pi}{6}$,

$y = \frac{1}{2}$

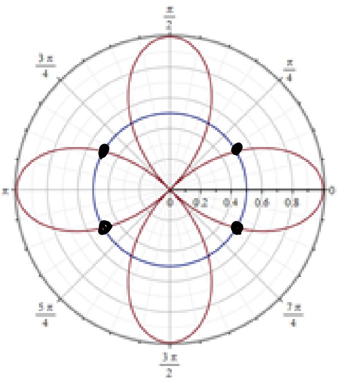
$y = -\frac{1}{2}$

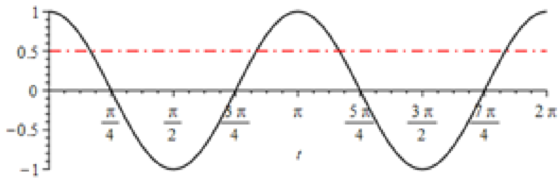


FACT $\cos(*)$ will intersect $r = \frac{1}{2}$ (circle) whenever

$\cos(*) = \frac{1}{2}$ or $\cos(*) = -\frac{1}{2}$

$r = -\frac{1}{2}$ is ALSO that same circle!



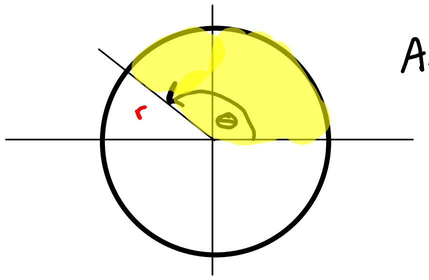


In §10.4, when we do areas inside/outside polar graphs, this can trip you up.

§10.4

Areas & Arc Lengths in Polar coords.

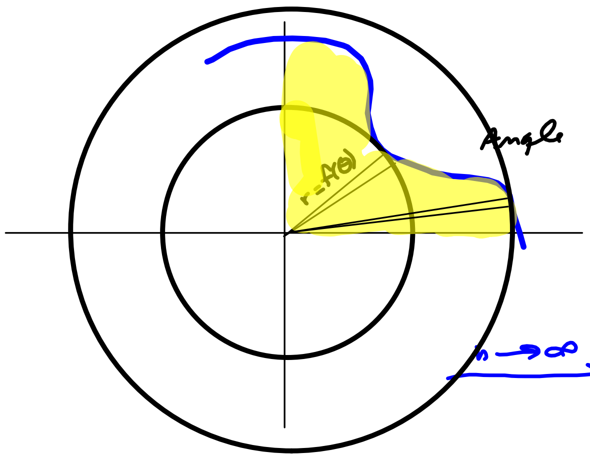
↳ It's all about area of a sector of a circle



Area is shaded yellow
$= \frac{1}{2} r^2 \theta$

{ πr^2 is area of circle &
{ Area is proportional to the angle
→ $\theta = 2\pi$, so $\pi r^2 = \frac{1}{2}(2\pi) r^2$
$= \frac{1}{2} \theta r^2$

Now, think of a very SMALL angle, say $d\theta$



Angle is $d\theta = \Delta\theta$

shaded Area is approximated

$$\sum_{k=1}^{n} \frac{1}{2}(f(\theta))^2 d\theta$$

$\xrightarrow{n \to \infty}$ $\frac{1}{2}\int_0^{\frac{\pi}{2}} f(\theta)^2 d\theta$

$$= \frac{1}{2}\int_a^b r^2 d\theta$$

Arc Length in Polar Coordinates.

$$L = \int_a^b ds = \int_a^b \sqrt{1+(f'(x))^2}\,dx \quad \text{where } y=f(x)$$

Parametric Form : $x = x(t),\ y = y(t) \Rightarrow$

$$L = \int_\alpha^\beta \sqrt{(x')^2 + (y')^2}\,dt$$

$$\left.\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \end{aligned}\right\} \text{From } r = f(\theta) \Rightarrow r' = f'(\theta)$$

So using the SAME PARAMETRIC FORM of arc length, using θ as the parameter.

$$x' = \frac{d}{d\theta}\left[r\cos\theta\right] = r'\cos\theta - r\sin\theta$$

$$y' = \frac{d}{d\theta}\left[r\sin\theta\right] = r'\sin\theta + r\cos\theta$$

$$\Rightarrow (x')^2 + (y')^2 = (r')^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ (r')^2\sin^2\theta + 2rr'\cos\theta\sin\theta + r^2\cos^2\theta$$

$$= (r')^2 + r^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

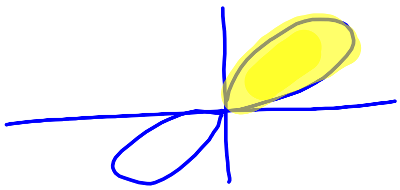
$$= (f'(\theta))^2 + f(\theta)^2$$

Polars :

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$$

$r^2 = \sin(2\theta)$ §10.4

Area of one lobe



$A = \frac{1}{2}\int_a^b r^2\,d\theta$

$= \frac{1}{2}\int_a^b \sin(2\theta)$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta$

$= \frac{1}{4}\left[-\cos(2\theta)\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{4}\left[-(1) - (-1)\right]$

$= \frac{1}{4}$

$\sin(2\theta) = 0$

$2\theta = 0, \pi, 2\pi, 3\pi, \cancel{4\pi}$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \cancel{2\pi}$

$\frac{1}{4}\left[-\cos(\pi) - (-\cos(0))\right]$

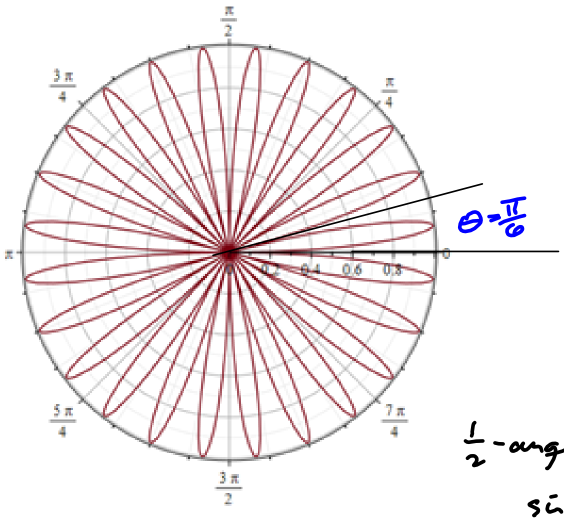
$= \frac{1}{4}\left[-(-1) - (-1)\right] = \frac{1}{2}$

10.4 #7

Area inside one loop of the curve. $r = \sin(12\theta)$

period: $12\theta = 2\pi$

$\theta = \frac{\pi}{6}$ = period.



$\frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2(12\theta)\,d\theta$ → what?!

$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos(24\theta))\,d\theta$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$\frac{1}{2}$-angle:

$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos(u)}{2}}$

$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1+\cos(u)}{2}}$

$\pm$ depends on quadrant.

$= \frac{1}{4}\int_0^{\frac{\pi}{6} \to \frac{\pi}{12}} d\theta - \frac{1}{4}\cdot\frac{1}{24}\int_0^{\frac{\pi}{6}} \cos(24\theta)\cdot 24\,d\theta$

$= \frac{1}{4}\left[\frac{\pi}{6 \to 12}\right] - \frac{1}{4\cdot 24}\cancel{\left[\sin(24\theta)\right]_0^{\frac{\pi}{6 \to 12}}} = \frac{\pi}{24 \to 48}$, idiot.

I'm off by a factor of 2!

WebAssign says $\frac{\pi}{48}$

$\sin(12\theta) = 0$ $\qquad 0 \le \theta \le 2\pi$

$12\theta = 0, \pi, \ldots$ $\qquad 0 \le 12\theta \le 24\pi$

$\theta = 0, \frac{\pi}{12}, \ldots$

Yup. 0 to $\frac{\pi}{12}$, not $\frac{\pi}{6}$.