§10.1 Parametric Eq'n Basics

(Last few exercises are simulations to play with.

$x = f(t),\ y = g(t)$

$x = \cos(t),\ y = 3\sin(t)$

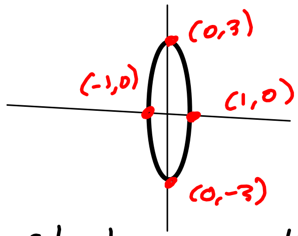
Note: $\frac{y}{3} = \sin(t) \longrightarrow$

$x^2 + \left(\frac{y}{3}\right)^2 = \cos^2(t) + \sin^2(t) = 1$

We're on the unit circle!

$x^2 + \left(\frac{y}{3}\right)^2 = 1$

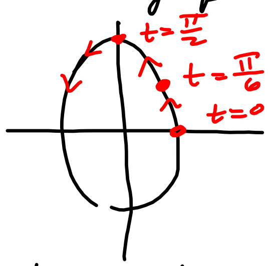
$x^2 + \frac{y^2}{9} = 1$ is an ellipse



Not a function, but a nice graph.

Parametrizing w.r.t. t gives the graph an orientation

t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{2}$
$\frac{\pi}{2}$	0	3



See Spreadsheet for today's talk in the notes.

Increasing t induces motion counter-clockwise around the ellipse.

Vector-Valued Functions with parameter t.

$\bar{r}(t) = \langle t, \sin(t), \cos(t) \rangle$

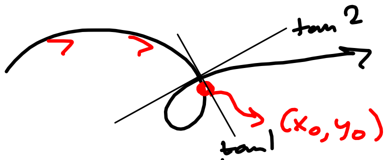
$\langle x\ ,\ y\ ,\ z \rangle$

a corkscrew

We'll find tangents, arc length, work done on a charged particle in a magnetic/electric field.

Retro-Rocket Thrust in 3-D!!!

Parametric Curves can cross themselves



(x_0, y_0) achieved @ more than one value of t, so 2 different tangents @ (x_0, y_0).

Depends on value of t.

10.1 Exercise:

$x = 1 - t^2$, $y = t - 4$ $\quad -2 \le t \le 2$

Eliminate parameter.
Get 2 functions $y = y(x)$.

$\Rightarrow$

$y + 4 = t \Rightarrow$

$x = 1 - (y+4)^2$

$\Rightarrow$

$x - 1 = -(y+4)^2 \Rightarrow$

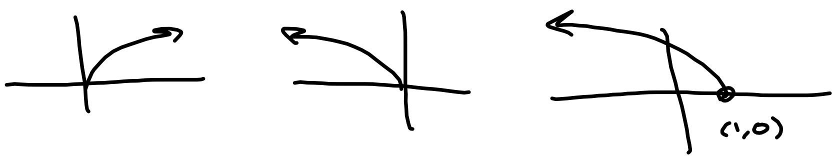
$(y+4)^2 = 1 - x$

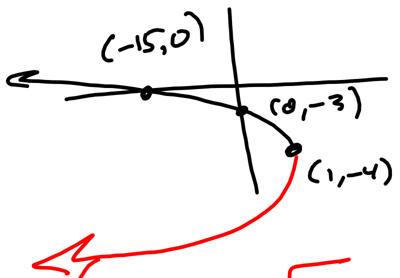
$y + 4 = \pm\sqrt{1-x}$

$\Rightarrow \quad y = \pm\sqrt{1-x} \; - 4$

$\sqrt{1-x} = \sqrt{-(x-1)}$

$\sqrt{x} \rightarrow \sqrt{-x} \rightarrow \sqrt{-(x-1)} \rightarrow \sqrt{-(x-1)} - 4$





√h.3 .3 $- \sqrt{1-x} \; - 4$

$\sqrt{1-x} \; - 4 = 0$

$\sqrt{1-x} = 4$

$1 - x = 16$

$-x = 15$

$x = -15 \rightarrow (-15, 0)$

WebAssign solved for x

$x = 1 - t^2, \quad y = t + 4 \implies y - 4 = t$

$\implies x = 1 - (y-4)^2 = 1 - (y^2 - 8y + 16)$

$= 1 - y^2 + 8y - 16$

$x = -y^2 + 8y - 15$ is what WebAssign wanted

is A cartesian eq'n of the circle.

Solving for y gives 2 eq'ns.

$y = +\sqrt{\quad}$

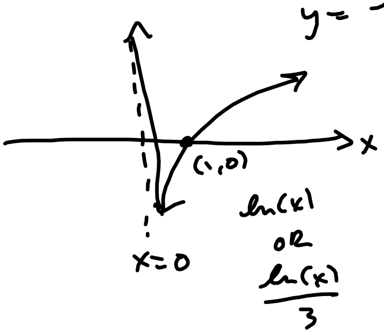
$y = -\sqrt{\quad}$

$x = e^{3t}$, $y = t + 4$ Elim. Param.

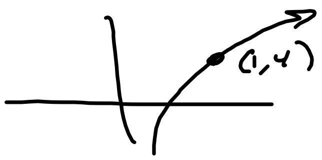
$\ln(x) = 3t$

$t = \frac{\ln(x)}{3} \Rightarrow$

$y = \frac{\ln(x)}{3} + 4$



$y = \frac{\ln(x)}{3} + 4$ (up 4 from previous)



§10.2 Calculus w/ Parametric Eq'ns

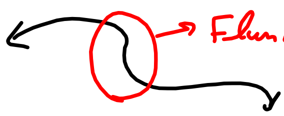
§ $x = f(t),\ y = g(t)$

Tangents to the curve. Assuming y is locally some $h(x)$ $= h(x(t))$

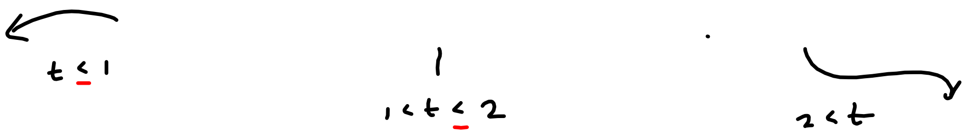
$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \longrightarrow \qquad \frac{d}{dt}\left[h(x(t))\right] = \frac{dh}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$, provided $\frac{dx}{dt} \neq 0$. NOTE: we didn't eliminate the parameter.

Remember. Parametric Curve might double-back on itself.



Flunks vertical line test,
But, for short stretches...



$\boxed{E}$ Find tangent line to
$x = \cos(t)$, $y = 3\sin(t)$ @ $t = \frac{\pi}{3}$

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{dy}{dt} = 3\cos(t)$$

$$\longrightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \left.\frac{3\cos(t)}{-\sin(t)}\right|_{t=\frac{\pi}{3}} = 3\cot\left(\frac{\pi}{3}\right) = -\frac{3}{\sqrt{3}} = m_{tan}$$

$$= -\frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}} = -\sqrt{3}$$

$$y = m_{tan}(x - x_0) + y_0$$

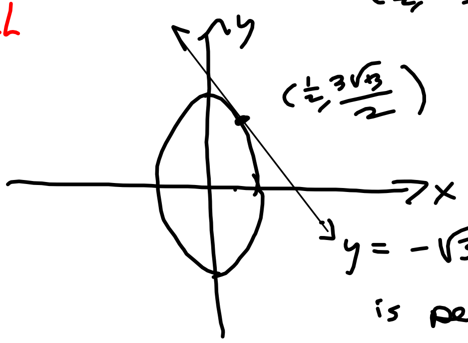
$$x_0 = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2} = y_0$$

$$y = -\sqrt{3}\left(x - \frac{1}{2}\right) + \frac{3\sqrt{3}}{2}$$

$\left(\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$

Hand Sketch

SEE MAPLE for syntax & better graphs.



$y = -\sqrt{3}\left(x - \frac{1}{2}\right) + \frac{3\sqrt{3}}{2}$ is perfect.

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \quad \text{using same idea \& previous work}$$

Recall $x = \cos(t)$, $y = 3\sin(t)$

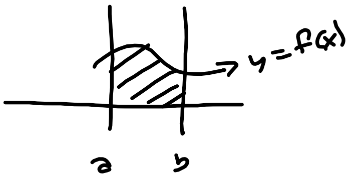
$$\frac{dy}{dx} = -3\cot(t) \quad \text{\& so}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\overbrace{3\csc^2(t)}^{\frac{d}{dt}\left[\frac{dy}{dx}\right]}}{\underbrace{-\sin(t)}_{\frac{dx}{dt}}} = -3\csc^3(t)$$

$$\frac{d}{dx}\left[\sim\right] = \frac{\frac{d\sim}{dt}}{\frac{dx}{dt}}$$

Area

Recall $\text{Area} = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$

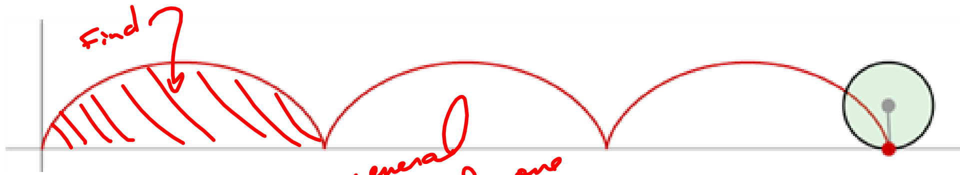


$x = f(t)$ $\qquad y = g(t)$

$dx = f'(t)\,dt$

EXAMPLE 3 Area under 1 arch of the cycloid.

$x = r(\theta - \sin\theta)\,,\ y = r(1-\cos\theta)$ Assume r = constant.



In general $\qquad$ for one arch.

$$\underbrace{\int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta}_{\text{Change of variables.}} = r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$\frac{dx}{dt} = r(1-\cos\theta)$$

$$= r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta$$

$$= r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right)d\theta$$

$$= r^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\right)d\theta$$

$$= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

$u = 2\theta$

$du = 2\,d\theta$

$$= \frac{3}{2}\cdot 2\pi r^2$$

$$= \boxed{3\pi r^2} = \text{Area}$$

I need to say something more/better about change-of-variable step.

Recall Arc Length for $y = f(x)$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx, \quad \& \ \sqrt{1+(f'(x))^2}\,dx$$

= arc length increment.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\ \frac{dx}{dt}\cdot dt$$

$$= \sqrt{\left(1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2\right)\left(\frac{dx}{dt}\right)^2}\ dt$$

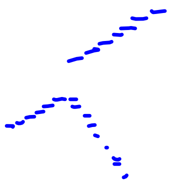
$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = ds \text{ for parametrics}$$

So, Arc Length $= L = \int_\alpha^\beta \sqrt{(x')^2+(y')^2}\,dt$

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}$$

Polynomials dense in cts functs. → Taylor Polynomials.

& Step functions are, too



My hangup:
why do I need
ds, when dx should work
in the limit?

You <u>Do Need</u> ds, you D.S.

You DO get different results when
you use $\int * \, dx$ -vs- $\int * \sqrt{1+f'^2} \, dx$

$\sqrt{1+x^2} \xrightarrow{x \to \infty} \sqrt{x^2}$ (more or less)

$= x$ if $x \geq 0$

Very Easy to set up.
A pain in the neck
to crank out all the
way. UGLY
INTEGRALS!

why not $\int f'(x) \, dx$ instead
of $\int \sqrt{1+f'(x)^2} \, dx$?
They're different!

I emphasize writing the integrals.

Surface Area of solids of revolution:

Basically the same as before

Recall $S.A. = 2\pi \int y \, ds$

Now, it's $= 2\pi \int y \, ds$, where

$$\boxed{ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt},$$

instead of one of

$\sqrt{1 + x'^2}\, dy$

or $\sqrt{1 + y'^2}\, dx$

We have a slightly different arc-length increment, ds.