$\sqrt{ax^2+bx+c}$ → $\sqrt{d^2-u^2}$, $\sqrt{u^2-d^2}$, $\sqrt{u^2+d^2}$

d = constant
u = variable

$$\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\int \frac{dx}{x^2+3x+2} = \int \frac{A\,dx}{x+1} + \int \frac{B\,dx}{x+2} = A\ln|x+1| + B\ln|x+2| + C$$

Owie! Sweet!

$$\cancel{(x+1)}\cancel{(x+2)}\left(\frac{1}{\cancel{x^2+3x+2}}\right) = \frac{\cancel{(x+1)}(x+2)\,A}{\cancel{x+1}} + \frac{(x+1)\cancel{(x+2)}\,B}{\cancel{x+2}} \Longrightarrow$$

(x+1)(x+2)

$1 = A(x+2) + B(x+1)$ is true for all real numbers for which it is defined. Trick:

$x=-1$:

$1 = A(-1+2) = \boxed{A=1}$

$x=-2$:

$1 = B(-2+1) = -B = 1 \Longrightarrow \boxed{B=-1}$

So we have

$$\int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$$

$$\frac{x^2-1}{x^3+x^2+x} = \frac{x^2-1}{x(x^2+x+1)}$$

$a=1=b=c$

$b^2-4ac = 1^2-4(1)(1) = -3 < 0$

$\Rightarrow x^2+x+1$ is "irreducible" over the real #s.

$$= \frac{x^2-1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$\Rightarrow x^2-1 = Ax^2+Ax+A+Bx^2+Cx$

EQUATING POWERS:

$\Rightarrow x^2 = Ax^2+Bx^2$

$1 = A+B$

$A = 1-B$

$\Rightarrow B = 1-A = 1-(-1) = 2$

$B=2$

$0 = Ax + Cx$

$0 = A+C$

$\boxed{A=-C}$

$\Rightarrow \boxed{C=1}$

$\boxed{-1=A}$

$$x^n + x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$$

$x^2-1 = (x-1)(x+1)$

$x^3-1 = (x-1)(x^2+x+1)$

$x^n-1 = (x-1)(x^{n-1}+x^{n-2}+\ldots+x^2+x+1)$

$$\frac{A}{x} + \frac{Bx+C}{x^2+x+1} = \frac{-1}{x} + \frac{2x+1}{x^2+x+1}$$

To integrate

$$-\int \frac{1}{x}dx + \int \frac{2x+1}{x^2+x+1}dx \qquad u = x^2+x+1$$
$$du = (2x+1)dx$$

$$= -\ln|x| + \ln|x^2+x+1| + C$$

What if it were $\int \frac{2x+7}{x^2+x+1}dx$?

$$\int \frac{2x+1}{x^2+x+1}dx + \int \frac{6}{x^2+x+1}dx$$

$$\int \frac{6}{x^2+x+1}dx = \int \frac{6\,dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

Let $u = x+\frac{1}{2} \Rightarrow du = dx \Rightarrow$

$$\int \frac{6du}{u^2+\frac{3}{4}} = \int \frac{6du}{u^2+(\frac{\sqrt{3}}{2})^2} = \frac{6\cdot 2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$\int \frac{du}{u^2+a^2}$$

#39 from my edition of book

$I = \int \frac{dx}{x\sqrt{x-1}}$

Try $\sqrt{x-1} = u$?

$x-1 = u^2$

$x = u^2+1$

$dx = 2u\,du$

$\Rightarrow I = \int \frac{2u\,du}{(u^2+1)u}$

Last Bit of Theory

Handling repeated factors:

$$\frac{5x+2}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$\frac{7}{(x+1)^3(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x-2)}$$

$7 = A(x+1)^2(x-2) + B(x+1)(x-2) + C(x-2) + D(x+1)^3$

$x=2$:

$7 = 27D$

$\boxed{D = \frac{7}{27}}$

$x=-1$:

$7 = -3C$

$\boxed{C = -\frac{7}{3}}$

$x=1$:

$7 = A(2)^2(-1) + B(2)(-1) + C(-1) + D(2)^3$

$= -4A - 2B - (-7) + (\frac{7}{27})(8)$

$-4A - 2B + 7 + \frac{56}{27} = 7$

$= -108A - 54B + 56 = 0$

$= -108A - 54B = -56$

$\Rightarrow -108A - 54B = -56$

$\boxed{108A + 54B = 56}$

$x=-2$:

$7 = A(-1)^2(-4) + B(-1)(-4)$
$+ C(-4) + D(-1)^3$

$= -4A + 4B - 4C - D$

$= -4A + 4B - 4(-7) - \frac{7}{27} = 7$

$= -4A + 4B + 28 - \frac{7}{27} = 7$

$\Rightarrow -4A + 4B = \frac{7}{27} - 21$

$-108A + 108B = 7 - 567 = -560$

$108A - 108B = 560$

$\boxed{27A - 27B = 140}$

27
21
27
540
567

2 | 108
2 | 54
3 | 27
3 | 9
3

2 | 560
2 | 280
2 | 140
2 | 70
5 | 35
7

$\boxed{108A + 54B = 56}$ $\boxed{27A - 27B = 140}$

Solve this system of 2 eqns in 2 unknowns

$$\begin{array}{rl} & 27A - 27B = 140 \\ + & 54A + 27B = 28 \\ \hline & 81A = 168 \end{array}$$

$$\boxed{A = \frac{168}{81}}$$

$$27A - 27B = \cancel{27}^{\,1}\left(\frac{\cancel{168}^{\,56}}{\cancel{81}_{\,\cancel{9}\,\cancel{3}\,1}}\right) - 27B = 140$$

$$\Longrightarrow 56 - 27B = 140$$

$$-27B = 84$$

$$\boxed{B = -\frac{84}{27}}$$