Basic Toolkit for Section 7.2 - Trigonometric Integrals

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

Power Reduction Formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Derivatives

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

Integration Formulas

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

See **Strategy for Evaluating** $\int \sin^m x \cos^n x \, dx$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

The trick:
when it's $u=\cos\theta$, then
$du=-\sin\theta\, d\theta$

$u=\sec\theta$
$du=\sec\theta\tan\theta\, d\theta$

$u=\tan\theta$
$du=\sec^2\theta\, d\theta$

$$\int \sin^5(x)\,dx = \int \sin^4(x)\sin(x)\,dx = \int \left(\sin^2(x)\right)^2 \sin(x)\,dx$$

$$= \int \left(1-\cos^2(x)\right)^2 \sin(x)\,dx = \int \left(\cos^2(x)-1\right)^2 \sin(x)\,dx$$

$$= \int \left(\cos^4(x) - 2\cos^2(x) + 1\right) \sin(x)\,dx$$

$$= \int \cos^4(x)\sin(x)\,dx - 2\int \cos^2(x)\sin(x)\,dx + \int \sin(x)\,dx$$

$u=\cos(x)$
$du=-\sin(x)\,dx$

$u=\cos(x)$
$du=-\sin(x)\,dx$

I'm engineering the "$-\sin(x)\,dx$" by changing the signs of the 1st 2 integrals

$$= -\int u^4\,du + 2\int u^2\,du - \cos(x) + C$$

$$= -\frac{\cos^5(x)}{5} + \frac{2\cos^3(x)}{3} - \cos(x) + C$$

$$-\frac{\cos(x)^5}{5} + \frac{2\cos(x)^3}{3} - \cos(x) + C$$

$$\int \cos^4(x)\,dx = \int \left(\cos^2(x)\right)^2 dx$$

$$= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \int \left(\frac{\cos(2x)+1}{2}\right)^2 dx$$

$$= \int \frac{\cos^2(2x) + 2\cos(2x) + 1}{4}\,dx$$

$$= \frac{1}{4}\int \left(\frac{1+\cos(4x)}{2} + 2\cos(2x) + 1\right) dx$$

$$= \frac{1}{8}\int (1+\cos(4x))\,dx + \frac{1}{4}\int 2\cos(2x)\,dx + \frac{1}{4}\int dx$$

$$= \frac{1}{8}\int dx + \frac{1}{8}\cdot\frac{1}{4}\int \cos(4x)\,4dx + \frac{1}{4}\int \cos(2x)\cdot 2dx + \frac{1}{4}\int dx$$

$u = 4x$, $du = 4dx$

$u = 2x$, $du = 2dx$

$$= \frac{3}{8}\int dx + \frac{1}{32}\int \cos(u)\,du + \frac{1}{4}\int \cos(v)\,dv$$

$$= \frac{3}{8}x + \frac{1}{32}(-\sin(u)) + \frac{1}{4}(-\sin(v)) + C$$

$$= \frac{3}{8}x - \frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + C$$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$5x^2 + 7x + 11$

$= 5(x^2 + \frac{7}{5}x \qquad) + 11$

$= 5(x^2 + \frac{7}{5}x + (\frac{7}{10})^2)$

$+11 - 5(\frac{49}{100})$

$= 5(x+\frac{7}{10})^2 + \frac{171}{20}$

$\frac{220}{20} - \frac{49}{20}$

$= \frac{171}{20}$

Recall $\frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{-\sin(x)\,dx}{\cos(x)} = -\int \frac{du}{u} = -\ln|u| + C$$

$u = \cos(x)$
$du = -\sin(x)dx$

$$= -1 \ln|\cos(x)| + C$$

$$= -\ln\left(|\cos(x)|^{-1}\right) + C = -\ln\left|(\cos(x))^{-1}\right| + C = -\ln|\sec(x)| + C$$

$$\boxed{\int \tan(x)dx = \ln|\sec(x)| + C}$$

$$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{du}{u} = \ln|u| + C$$

$u = \sec(x) + \tan(x)$

$\Rightarrow du = (\sec(x)\tan(x) + \sec^2(x))\,dx$

$$\boxed{\begin{aligned} &= \ln|\sec(x) + \tan(x)| + C \\ &= \int \sec(x)\,dx \end{aligned}}$$

10. 0/1 points

By using integration by parts, one can prove the reduction formula.

$$\int (\ln(x))^n\,dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\,dx$$

Use this formula to find $8\int (\ln(x))^3\,dx$. (Use C for the constant of integration.)

$C + 8x(\ln(x))^3 - 24x(\ln(x))^2 + 48x\ln(x) - 48x$

$$\int (\ln(x))^3 dx = x(\ln(x))^3 - 3\int (\ln(x))^2 dx$$

$$= x(\ln(x))^3 - 3\left[x\ln(x)^2 - 2\int \ln(x)\,dx\right]$$

$$= x(\ln(x))^3 - 3x(\ln(x))^2 + 6\int \ln(x)dx$$

$u = \ln(x)$ $dv = dx$

$du = \frac{1}{x}dx$ $v = x$

$$= x(\ln(x))^3 - 3x(\ln(x))^2 + 6\left[x\ln(x) - \int x\cdot\frac{1}{x}dx\right]$$

$$= x(\ln(x))^3 - 3x(\ln(x))^3 + 6x\ln(x) - 6\int dx$$

$$= x(\ln(x))^3 - 3x(\ln(x))^3 + 6x\ln(x) - 6x + C$$

$C + 8x(\ln(x))^3 - 24x(\ln(x))^2 + 48x\ln(x) - 48x$

off by a factor of 8
(missed the 8 out front)

Product-to-Sum Formulas:

$$\int \sin(17x)\cos(11x)\,dx$$

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$$= \frac{1}{2}\int \left(\sin(6x) + \cos(28x)\right)dx = \frac{1}{2}\left[-\frac{1}{6}\cos(6x) + \frac{1}{28}\sin(28x)\right] + C$$

$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

$\frac{d}{dx}[\tan(x)] = \sec^2(x)$

You're always looking for "a du"

$\sec^2(x)\,dx$

$\sec(x)\tan(x)\,dx$

$\int \sec^4(x)\,dx$ $\qquad$ $\int \sec^4(x)\tan^2(x)\,dx$ $\qquad$ $\int \sec^4(x)\tan^3(x)\,dx$

Same $\qquad$ Same

$\tan^2\theta + 1 = \sec^2\theta$

$= \int (\tan^2(x)+1)\sec^2(x)\,dx$

$= \int \underbrace{\tan^2(x)\sec^2(x)\,dx}_{u^2 du} + \int \sec^2(x)\,dx$

$= \frac{1}{3}\tan^3(x) + \tan(x) + C$

$\int \sec^3(x)\tan^3(x)\,dx$ looks tough

$= \int \sec^2(x)\sec(x)\tan^2(x)\,dx$ $\qquad$ UGH! $\qquad$ Integration by Parts

$= \int \sec^2(x)\tan(x)\,\underline{\sec(x)\tan(x)\,dx}$ $\qquad$ $\int \sec^{ODD}\tan^{even}$ sucks!

$\int \sec^3(x)\tan^3(x)\,dx = \int \sec^2(x)\tan^2(x)\sec(x)\tan(x)\,dx$

$= \int \sec^2(x)(\sec^2(x) - 1)\sec(x)\tan(x)\,dx$

$= \int \sec^4(x)\sec(x)\tan(x)\,dx - \int \sec^2(x)\sec(x)\tan(x)\,dx$ $\qquad$ $u^n du$ situations!

$= \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

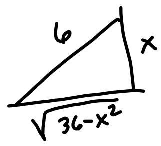
§7.5 #10

10. 0/1 points

Evaluate the integral. (Use C for the constant of integration.)

$$\int \sqrt{\frac{6+x}{6-x}}\,dx$$

$$\sqrt{\frac{6+x}{6-x}\cdot\frac{6+x}{6+x}} = \frac{\sqrt{(6+x)^2}}{\sqrt{36-x^2}} = \frac{|6+x|}{\sqrt{36-x^2}}$$, but by original



statement $\sqrt{6+x}$ assumes that $6+x \geq 0$, & $\sqrt{6-x} \Rightarrow$

$6-x \geq 0 \Rightarrow 6 \geq x$

We don't need $|6+x|$, so

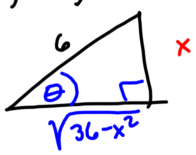
$u = 36-x^2$
$du = -2x\,dx$

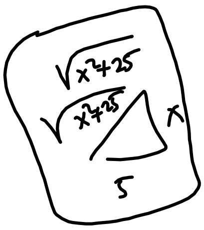
$$\int \frac{6+x}{\sqrt{36-x^2}}\,dx = \int \frac{6}{\sqrt{36-x^2}}\,dx - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{36-x^2}}$$

Either Trig Sub or See inverse trig section. (arcsine sitch)

=

$\sqrt{36-x^2}$ looks pythagorean





$x = 6\sin\theta$
$dx = 6\cos\theta\,d\theta$

$\frac{x}{6} = \sin\theta$

$x = 6\sin\theta$

$$\int \frac{6}{\sqrt{36-x^2}}\,dx = \int \frac{6}{\sqrt{36-36\sin^2\theta}}\,dx$$

$$= \int \frac{\cancel{6}}{\cancel{6}\sqrt{1-\sin^2\theta}}\,6\cos\theta\,d\theta = \int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$

$$= \int \frac{\cos\theta\,d\theta}{|\cos\theta|} = \int d\theta \text{ if } \cos\theta > 0$$

$$\& = -\int d\theta \text{ if } \cos\theta < 0$$

Assume $0 < \theta < \frac{\pi}{2}$

$\Rightarrow \cos\theta > 0$

$$\Rightarrow \int \frac{6}{\sqrt{36-x^2}}\,dx = \int 6\,d\theta = 6\theta$$

$x = 6\sin\theta$

$$= 6\arcsin\left(\frac{x}{6}\right) + C$$

$\frac{x}{6} = \sin\theta$

$\theta = \arcsin\left(\frac{x}{6}\right)$