MAT 202
100 Points

Test 3, Spring, 2020
Covers Chapters 8 and 9

Name KEY

Do all your work and put all your answers WITH your work, CIRCLED, on the white paper provided. All I want on this sheet is your NAME! Work up to 4 **Bonus** problems.

1. (15 pts) Write the integral for the arc length of the graph of the function $y=\sqrt{4-x^2}$ from $x=-1$ to $x=2$.

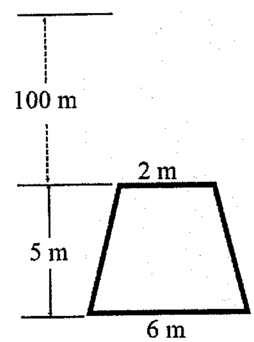
2. (5 pts **Bonus**) Find the exact arc length from #1.

3. Write the integral for the surface area obtained by revolving the graph of $x=y^2+2$ for $2\le x\le 11$ about the ...

   a. (10 pts) ... y-axis.

   b. (10 pts) ... x-axis.

4. (5 pts **Bonus**) Give an exact answer for #3b. Hint: I found this one to be easier if I inverted the $x=g(y)$ to get $y=g^{-1}(x)$ and used that formulation for the integral.



5. (10 pts) A gate at the bottom of a hydro-electric dam is as shown in the picture. Recall that the density of water is $1000\frac{kg}{m^3}$ and the acceleration due to gravity is $9.8\frac{m}{s^2}$. Write the integral giving the hydrostatic force on the submerged plate.

6. (5 pts) Find the force on the plate, to the nearest Newton.

7. (10 pts) The growth rate of a trout population in a beaver pond is proportional to the population, itself. If they stock the pond with 20 trout, in 5 years, there are 500 trout. How many trout will be in the pond in 100 years, if nothing happens to slow the rate of population growth?

8. (5 pts) What are the problems with the population model in #7, if any?

9. (10 pts) Solve the differential equation $\frac{dp}{dt}=t^2p+t^2-p-1$. Hint: This equation is separable. To see this, factor the right-hand-side, by grouping.

10. (10 pts) Solve the differential equation $2xy'+y=2\sqrt{x}$

11. (15 pts) $y''-3y'-10=0$

12. (5 pts **Bonus**) Evaluate $\int_e^{\infty}\frac{dx}{x(\ln(x))^4}$

13. (5 pts **Bonus**) Evaluate $\int x^2\ln\left(\sqrt[5]{x^4}\right)dx$

14. (5 pts **Bonus**) If $y=(1+x)^{\frac{3}{x}}$, what is $\frac{dy}{dx}$?

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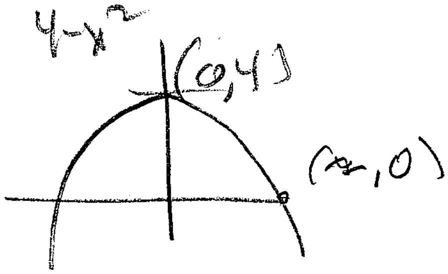
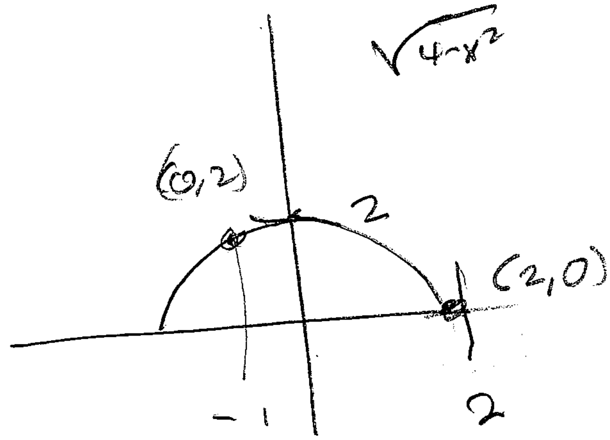
(1) (15 pts) $y=\sqrt{4-x^2}$, $x=-1$ to $x=2$





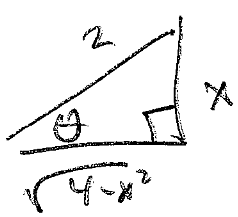
$y' = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$

$= -\frac{x}{\sqrt{4-x^2}} \longrightarrow$

$(y')^2 = \frac{x^2}{4-x^2} \longrightarrow 1+(y')^2 = \frac{4-x^2+x^2}{4-x^2} = \frac{4}{4-x^2}$

$\Rightarrow \int_{-1}^{2} ds = \int_{-1}^{2}\sqrt{\frac{4}{4-x^2}}\,dx = \int_{-1}^{2}\frac{2}{\sqrt{4-x^2}}\,dx$

(2) (5B)



$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x = 2\sin\theta = -1 \longrightarrow \theta = -\frac{\pi}{6}$

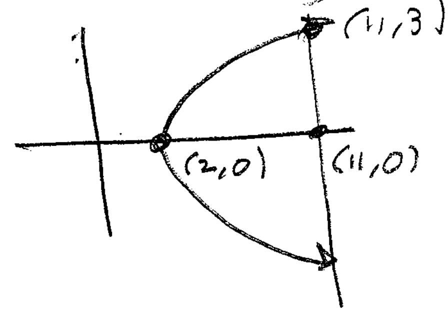
$\sqrt{4-x^2} = 2\cos\theta$, on $[-\frac{\pi}{6}, \frac{\pi}{2}]$ $x = 2\sin\theta = 2 \Rightarrow \theta = \frac{\pi}{2}$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2}{2\cos\theta}\cdot 2\cos\theta\,d\theta = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta = 2\left[\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right] = \boxed{\frac{4\pi}{3}}$

3 a 10 pts $x = y^2 + 2$, $2 \le x \le 11$

Surface Area of solid obtained by revolving

$x = y^2 + 2$ about the y-axis :



$$A = 2\pi \int_a^b x \, ds$$

$x' = 2y$

$(x')^2 = 4y^2$

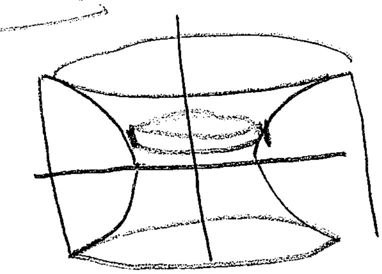
$y^2 + 2 = 2$ $\quad y^2 + 2 = 11$

$y^2 = 0$ $\quad y^2 = 9$

$y = 0$ $\quad y = \pm 3 \rightarrow y = 3$

$$\rightarrow \boxed{A = 2\pi \int_0^3 (y^2+2)\sqrt{4y^2+1} \, dy} \text{ TIMES } 2!$$

OR $A = 2\pi \int_2^{11} x \, ds = 2\pi$



$x = y^2 + 2$ $\quad$ use $+\sqrt{x-2} = y$

$x - 2 = y^2$ $\quad \rightarrow y' = \frac{1}{2}(x-2)^{-\frac{1}{2}} \rightarrow (y')^2 = \frac{1}{4\sqrt{x-2}}$

$\pm\sqrt{x-2} = y$

$$\rightarrow \boxed{A = \int_2^{11} x \sqrt{1 + \frac{1}{4\sqrt{x+2}}} \, dx} = \int_2^{11} 2\pi x \sqrt{\frac{4\sqrt{x-2} + 1}{\sqrt{x-2}}}$$

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3a cont'd $= 2\pi \int_2^{11} x \sqrt{\dfrac{4(x-2)+\sqrt{x-2}}{x-2}}\, dx$

Let $u = x-2$. Then $du = dx$ & $x = u+2$

$= 2\pi \int_{x=2}^{x=11} (u+2) \sqrt{\dfrac{4u+\sqrt{u}}{u}}\, du$

$= 2\pi \int_{x=2}^{x=11} \left(\sqrt{4u+\sqrt{u}}\, du + \dfrac{2\sqrt{4u+\sqrt{u}}}{u}\right) du$

ugh!

10 pts

3b Same question, about the x-axis

$A = 2\pi \int_a^b y\, ds = 2\pi \int_0^3 y\sqrt{4y^2+1}\, dy$

OR $2\pi \int y\, ds = 2\pi \int_2^{11} \sqrt{x-2} \sqrt{1+\left(\frac{1}{2}(x-2)^{-\frac{1}{2}}\right)^2}\, dx$

4 5B $= \dfrac{2\pi}{8} \int_0^3 (\sqrt{4y^2+1})(8y\, dy)$

$= \dfrac{\pi}{4}\left[\dfrac{2}{3}(4y^2+1)^{\frac{3}{2}}\right]_0^3 = \dfrac{\pi}{6}\left[37^{\frac{3}{2}} - 1\right]$

$= \boxed{\dfrac{\pi}{6}\left[37\sqrt{37} - 1\right]}$

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#3b Alternate Formulation

$$2\pi \int_a^b y\, ds = 2\pi \int_2^{11} \sqrt{x-2}\, \frac{\sqrt{4x-7}}{2\sqrt{x-2}}\, dx$$

$$y = (x-2)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x-2)^{-\frac{1}{2}}$$

$$ds = \sqrt{1+(y')^2}\, dx$$

$$= \sqrt{1+\frac{1}{4}(x-2)^{-1}}\, dx$$

$$= \sqrt{\frac{4(x-2)+1}{4(x-2)}}\, dx = \sqrt{\frac{4x-7}{4(x-2)}}\, dx$$

$$= \frac{\pi}{4} \int_2^{11} \sqrt{4x-7}\, dx = \frac{\pi}{4}\left[\frac{2}{3}(4x-7)^{\frac{3}{2}}\right]_2^{11}$$

$$= \frac{\pi}{6}\left[(4(11)-7)^{\frac{3}{2}} - (4(2)-7)^{\frac{3}{2}}\right]$$

$$= \frac{\pi}{6}\left[37^{\frac{3}{2}} - 1\right] = \frac{\pi}{6}\left[37\sqrt{37} - 1\right]$$

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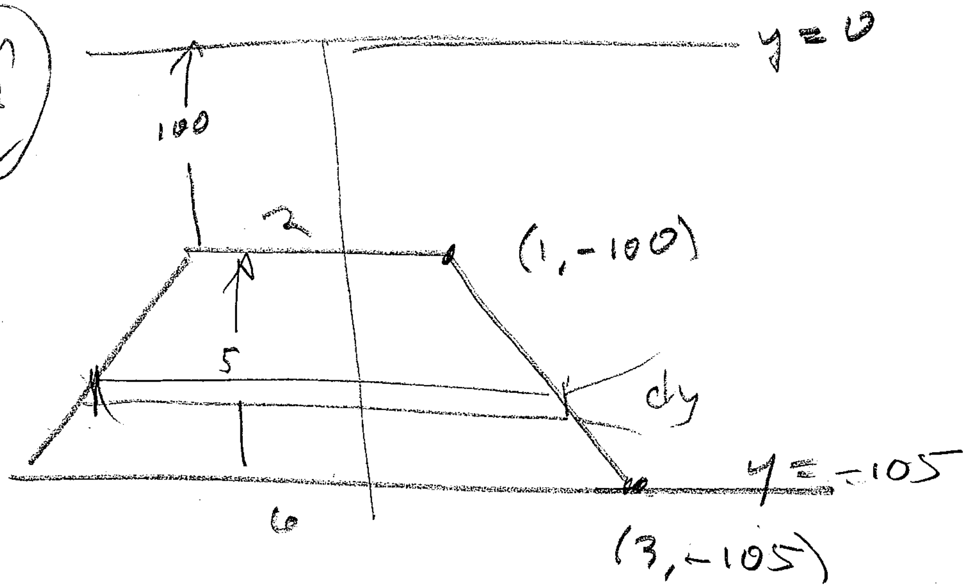
(5) (10 pts)



$$\frac{-105-(-100)}{3-1} = \frac{-5}{2} = m$$

$$y = -\frac{5}{2}(x-1) - 100 = -\frac{5}{2}x + \frac{5}{2} - \frac{200}{2} = -\frac{5}{2}x - \frac{195}{2}$$

Now, Area of dy-rectangle:

$$A = 2(x)\,dy \quad \& \quad y = -\frac{5}{2}x - \frac{195}{2} \Rightarrow$$

$$-\frac{5}{2}x = y + \frac{195}{2}$$

$$x = -\frac{2}{5}y - \frac{195}{5}, \text{ so}$$

Area of representative rectangle is

$$A = 2\left(-\frac{2}{5}y - \frac{195}{5}\right)dy$$

Force on that rectangle is $\left(1000\frac{kg}{m^3}\right)(-y)(A)\left(\frac{9.8m}{s^2}\right)$

$$= 9800(-y\ m)\left(-\frac{4}{5}y - \frac{390}{5}\right)(m^2)\,dy$$

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⇒ FORCE ON SUBMERGED PLATE IS:

$$F = \int_{-105}^{-100} 9800(-y)\left(-\frac{4}{5}y - \frac{390}{5}\right) dy$$

$$= \frac{9800}{5} \int_{-105}^{-100} (4y^2 + 390y)\, dy$$

$$= \frac{\overset{1960}{\cancel{9800}}}{\cancel{5}} \left[\frac{4}{3}y^3 + 195y^2\right]_{-105}^{-100}$$

$$= 1960\left[\frac{4}{3}(-100)^3 + 195(-100)^2 - \left(\frac{4}{3}(-105)^3 + 195(-105)^2\right)\right]$$

$$= 1960\left[-\frac{4}{3}(10^2)^3 + 195(10)^4 + \frac{4}{3}(105)^3 - 195(105)^2\right]$$

$$= 1960\left[-\frac{4}{3}(10^6) + \frac{4}{3}(105)^3 + 1950000 - 195(105)^2\right]$$

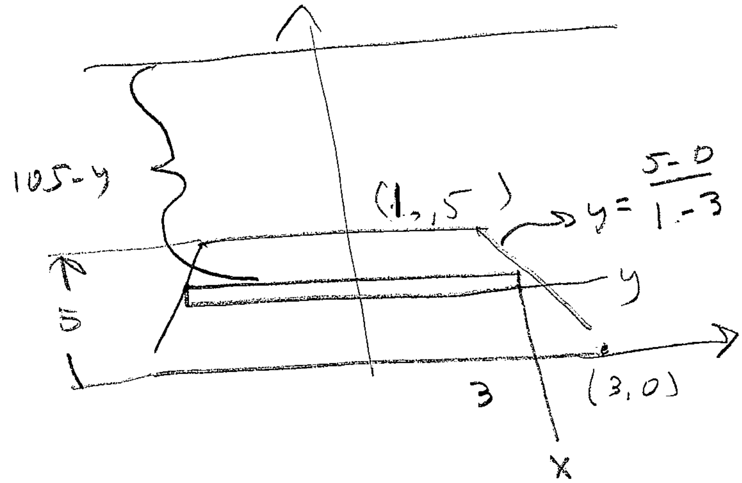
$$= \boxed{20{,}171{,}666.\overline{6}\ N}$$

$\frac{21}{4}$

$\overline{64}$

$(105)(-\frac{4}{5}y) + 630 + \frac{4}{5}y^2 - 6y$

$-84y + 630 + \frac{4}{5}y^2 - 6y = \frac{4}{5}y^2 - 90y + 630$



$y = \frac{5-0}{1-3}(x-3) + 0$

$= -\frac{5}{2}(x-3)$

$= -\frac{5}{2}x + \frac{15}{2}$

$2y = -5x + 15$

$2y - 15 = -5x$

$x = -\frac{2}{5}y + 3$

Area $= 2x\,dy = 2(-\frac{2}{5}y + 3)\,dy$

$= (-\frac{4}{5}y + 6)\,dy$

Force = Pressure · Area on rectangle

$= 9800(105-y)(-\frac{4}{5}y + 6)\,dy$

Add the Forces:

$9800\int_0^5 (105-y)(-\frac{4}{5}y+6)\,dy$

$= 9800\int_0^5 (\frac{4}{5}y^2 - 90y + 630)\,dy$

$= 9800\left[\frac{4}{15}y^3 - 45y^2 + 630y\right]_0^5 = 9800\left[\frac{4}{15}(5)^3 - 45(5)^2 + 630(5)\right]$

$= 9800\left[\frac{100}{3} - 1125 + 3150\right] = 9800\left[\frac{6175}{3}\right]$

$\boxed{= 20{,}171{,}666.\overline{6}\ N} \approx \boxed{20{,}171{,}667\ N}$

202 T2 (6)

7 10 pts

$P(t) = P_0 e^{kt}$ t = time, in years

$P(0) = 20 = P_0$

$P(t) = 20e^{kt}$

$P(5) = 500 = 20e^{k(5)}$

$\Rightarrow e^{5k} = \frac{500}{20} = 25$

$5k = \ln(25)$

$k = \frac{\ln(25)}{5}$

$\Rightarrow P(100) = 200e^{\left(\frac{\ln(25)}{5}\right)(100)}$

$= 20e^{20\ln(25)} = 20e^{\ln(25)^{20}} = 20e^{(\ln(5))^{40}}$

OR $20(5)^{40}$

$= 20(25)^{20} \approx 1.818989404 \times 10^{29}$

1.81,989,404 00,000,000,000,000,000,000.

182 OCTILLION!

Eve
4/7, 10:05 am
7151 Precision Drive,
Ft. Collins.

8 5B No population grows without some limits. Clearly, 181 OCTILLION ~~dont~~ won't even fit in the beaver pond.

202 $\sqrt{3}$ 7

9 10pts $p' = t^2 p + t^2 - p - 1$

$= t^2(p+1) - 1(p+1)$

$= (p+1)(t^2-1) \implies$

$$\frac{p'}{p+1} = t^2 - 1$$

$$\implies \int \frac{dp}{p+1} = \int (t^2-1)\, dt$$

$$\ln|p+1| = \frac{1}{3}t^3 - t + C$$

$$|p+1| = e^{\frac{1}{3}t^3 - t + C} = e^C e^{\frac{1}{3}t^3 - t} = |p+1|$$

Assume $p \geq -1$:

$$\boxed{p = e^C e^{\frac{1}{3}t^3 - t} - 1}$$

10 10 pts

$2xy' + y = 2\sqrt{x}$

$\rightarrow y' + \frac{1}{2x}y = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

I.F. $e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln|x| + C} = e^{C}e^{\ln\sqrt{x}}$

$= \hat{C}\sqrt{x}$. Let $\hat{C} = 1$.

Then $(\sqrt{x}y)' = \sqrt{x} \cdot \frac{1}{\sqrt{x}} = 1$

$\sqrt{x}\,y = x + C$

$\boxed{y = \sqrt{x} + \frac{C}{\sqrt{x}}}$

Check $y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}Cx^{-\frac{3}{2}}$

$2xy' + y = 2x\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}Cx^{-\frac{3}{2}}\right) + x^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$

$= x^{\frac{1}{2}} - \frac{C}{\sqrt{x}} + x^{\frac{1}{2}} + Cx^{-\frac{1}{2}} = 2x^{\frac{1}{2}} = 2\sqrt{x}$ ✓

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11 15 pts $y'' - 3y' - 10y = 0$

$D^2 - 3D - 10 = 0$

$(D-5)(D+2) = 0$

$\boxed{y = c_1 e^{5t} + c_2 e^{-2t}}$

12 5B $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^4} = \lim_{t\to\infty} \int_e^t (\ln x)^{-4}\left(\frac{1}{x}\,dx\right)$

$u = \ln x \quad du = \frac{1}{x}dx$

$\displaystyle = \lim_{t\to\infty}\left[\frac{(\ln(x))^{-3}}{-3}\right]_e^t = -\frac{1}{3}\left[\lim_{t\to\infty} \ln(t)^{-3} - \ln(e)^{-3}\right]$

$= -\frac{1}{3}\left[0 - (-1)\right] = \boxed{\frac{1}{3}}$

$\displaystyle\lim_{t\to\infty}\left[\frac{u^{-3}}{-3}\right]_1^t$

202 T3 10

13 5B $\int x^2 \ln\left(\sqrt[5]{x^4}\right) dx$

$u = \ln x \quad dv = x^2 dx$

$= \frac{4}{5}\int x^2 \ln x \, dx \qquad du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

$= \frac{4}{5}\left[uv - \int v\,du\right] = \frac{4}{5}\left[\frac{1}{3}x^3 \ln x - \int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}dx\right)\right]$

$= \frac{4}{15}x^3 \ln x - \frac{4}{15}\int x^2 dx = \frac{4}{15}x^3 \ln x - \frac{4}{15}\left(\frac{1}{3}x^3\right) + C$

$= \boxed{\frac{4}{15}x^3 \ln x - \frac{4}{45}x^3 + C}$

14 5B $y = (1+x)^{\frac{3}{x}} \longrightarrow$

$\ln y = \frac{3}{x}\ln(1+x) \longrightarrow$

$\frac{y'}{y} = -\frac{3}{x^2}\ln(x+1) + \left(\frac{3}{x}\right)\left(\frac{1}{x+1}\right) \longrightarrow$

$\boxed{y' = \left(-\frac{3}{x^2}\ln(x+1) + \frac{3}{x(x+1)}\right)(1+x)^{\frac{3}{x}}}$