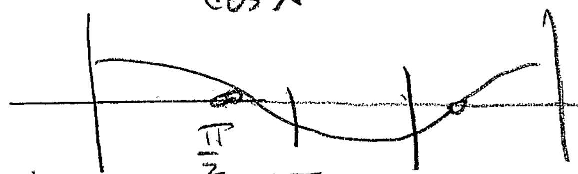
202 TAKE-HOME TEST 2

(1) 5 pts

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sqrt{1-\sin x}\, dx = I$$

Let $u = 1-\sin x$. Then $du = -\cos x\, dx$ ⟶ $\cos x$

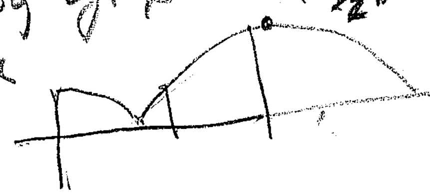


$$I = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{\sqrt{1-\sin x}\ (-\cos x\, dx)}{-\cos x}$$

Note $-\cos x = |\cos x|$ on $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$, b/c $\cos x < 0\ \forall x \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ by graph.

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{\sqrt{1-\sin x}}{\sqrt{\cos^2 x}} (-\cos x\, dx)$$

$(\frac{3\pi}{2}, 2)$



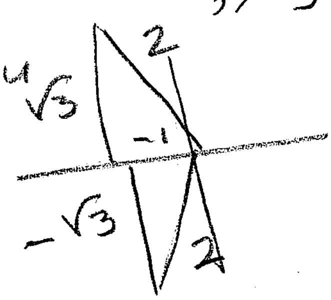
$\cos^2 x = 1-\sin^2 x$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{\sqrt{1-\sin x}}{\sqrt{(1-\sin x)(1+\sin x)}} (-\cos x\, dx)$$

Let $u = 1+\sin x$ then $du = \cos x\, dx$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+\sin x)^{-\frac{1}{2}} (-\cos x) dx$$

$\frac{2\pi}{3}, \frac{4\pi}{3}$



$$= -\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+\sin x)^{-\frac{1}{2}} (\cos x\, dx) = -\int_{x=\frac{2\pi}{3}}^{x=\frac{4\pi}{3}} u^{-\frac{1}{2}} du$$

$$= 2u^{\frac{1}{2}}\Big]_{x=\frac{2\pi}{3}}^{x=\frac{4\pi}{3}} = 2(1+\sin x)^{\frac{1}{2}}\Big]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= -2\left[\left(1+\sin\frac{4\pi}{3}\right)^{\frac{1}{2}} - \left(1+\sin\frac{2\pi}{3}\right)^{\frac{1}{2}}\right] = \left(\sqrt{1-\frac{\sqrt{3}}{2}} - \sqrt{1+\frac{\sqrt{3}}{2}}\right)(-2)$$

#1 Cnt'd Calculator will show this,

$$-2\left(\sqrt{1-\frac{\sqrt{3}}{2}}-\sqrt{1+\frac{\sqrt{3}}{2}}\right)$$

$$=2\sqrt{\frac{2+\sqrt{3}}{2}}-2\sqrt{\frac{2-\sqrt{3}}{2}}=\sqrt{2^2}\sqrt{\frac{2+\sqrt{3}}{2}}-\sqrt{2^2}\sqrt{\frac{2-\sqrt{3}}{2}}$$

$$=\sqrt{4\left(\frac{2+\sqrt{3}}{2}\right)}-\sqrt{4\left(\frac{2-\sqrt{3}}{2}\right)}=\sqrt{4+2\sqrt{3}}-\sqrt{4-2\sqrt{3}}$$

$$=\sqrt{(1+\sqrt{3})^2}-\sqrt{(1-\sqrt{3})^2}=1+\sqrt{3}-(\sqrt{3}-1)\boxed{=2.}$$

CHECK $(1+\sqrt{3})^2=1^2+2\sqrt{3}+\sqrt{3}^2=1+2\sqrt{3}+3=4+2\sqrt{3}$

$(1-\sqrt{3})^2=1^2-2\sqrt{3}+\sqrt{3}^2=1-2\sqrt{3}+3=4-2\sqrt{3}$

I wouldn't see that in a million years.

2 Simpson's, $n=10$, to approximate $\int_{-2}^{6}e^{-x^2}dx$

$\frac{b-a}{n}=\frac{6-(-2)}{10}=\frac{8}{10}=\frac{4}{5}$ Simpson's: $\frac{\Delta x}{3}[$

$$=\frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)$$
$$+2f(x_6)+4f(x_7)+2f(x_8)+4f(x_9)+f(x_{10})\Big]$$

$$=\frac{4}{15}\Big[e^{-(2)^2}+4e^{-(1.2)^2}+2e^{-(.4)^2}+4e^{-.4^2}+2e^{-1.2^2}+4e^{-2^2}$$
$$+2e^{-2.8^2}+4e^{-3.6^2}+2e^{-4.4^2}+4e^{-5.2^2}+e^{-6^2}\Big]$$

$$\approx\frac{4}{15}\Big[6.626804237\Big]\approx 1.767147796$$

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#2 cont'd

$$|E_{10}| \le \frac{K(6-(-2))^5}{180(10)^4} = \frac{8^5 K}{180(10)^4} = \frac{2^5 \cdot 4^5}{18 \cdot 10^5} K$$

$$= \frac{2^5 \cdot 4^5 K}{18 \cdot 2^5 \cdot 5^5} = \frac{4^5}{18 \cdot 5^5} K, \text{ where } K \ge |f^{(4)}(x)| \text{ on } [-2,6]$$

$\forall x \in [-2,6]$

FROM BELOW

$K = 12 \Rightarrow$

$f(x) = e^{-x^2}$

$|E_{10}| \le \frac{4^5}{18 \cdot 5^5} K \le \boxed{0.218453\overline{3}}$

$|E_{10}| \le$

$f^{(1)}(x) = -2xe^{-x^2}$

$f^{(2)}(x) = -2e^{-x^2} - 2x \cdot (-2x)e^{-x^2}$

$= -2e^{-x^2} + 4x^2e^{-x^2}$

$f^{(3)}(x) = 4xe^{-x^2} + 8xe^{-x^2} + 4x^2(-2xe^{-x^2})$

$= 12xe^{-x^2} - 8x^3e^{-x^2}$

$f^{(4)}(x) = 12e^{-x^2} + 12x(-2xe^{-x^2}) - 24x^2e^{-x^2} - 8x^3(-2xe^{-x^2})$

$= 12e^{-x^2} - 24x^2e^{-x^2} - 24x^2e^{-x^2} + 16x^4e^{-x^2}$

$= 12e^{-x^2} - 48x^2e^{-x^2} + 16x^4e^{-x^2}$

$f^{(5)}(x) = -24xe^{-x^2} - 96xe^{-x^2} - 48x^2(-2xe^{-x^2})$

$+ 64x^3e^{-x^2} + 16x^4(-2xe^{-x^2})$ SET 0

$= -120xe^{-x^2} + 160x^3e^{-x^2} - 32x^5e^{-x^2}$

$\Rightarrow x = 0, x \approx \pm .9585724645, \pm 2.020182840$ on $[-2,6]$

$|f^{(4)}(x)| \le 12 \equiv K$

3 5 pts Rieman Sum, same as ~~#~~ 2, but just right endpoints

$\Delta x \sum f(x_k)$ $\quad x_k = a + k\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{6-(-2)}{10} = \frac{8}{10} = \frac{4}{5}$

$x_k = -2 + \frac{4k}{5} = \frac{-10+4k}{5}$

$A \approx \frac{4}{5} \sum_{k=1}^{10} e^{-\left(\frac{4k-10}{5}\right)^2}$ Actual: 1.768308316

$= \frac{4}{5}\left[e^{-1.2^2} + e^{-.4^2} + e^{-.4^2} + e^{-1.2^2} + e^{-2^2} + e^{-2.8^2} + e^{-3.6^2} + e^{-4.4^2} + e^{-5.2^2} + e^{-6^2}\right]$

$\approx \frac{4}{5}\left[2.1968547760\right] \approx 1.757483807$

4 $\int_1^{\infty} \frac{x^2\,dx}{\sqrt[4]{x^{13}} + x^2 + 1}$ converges because it's ...

a essentially $\frac{x^2}{x^{13/4}} = \frac{1}{x^{\frac{13-8}{4}}} = \frac{1}{x^{\frac{5}{4}}}$ & $\frac{5}{4} > 1$

gives p-test for convergence.

b $\frac{x^2}{\sqrt[4]{x^{13}} + x^2 + 1} < \frac{x^2}{x^{13/4}} = x^{-\frac{5}{4}} = \frac{1}{x^{\frac{5}{4}}}$ ✓

$\int_1^{\infty} \frac{dx}{x^{5/4}}$ converges by p-tests

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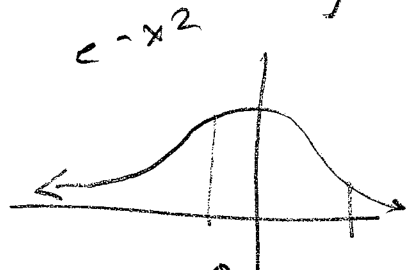
(3.2)

#3

Error Estimates for Right-Endpoint method have never really been discussed. By symmetry about y-axis, we have



$R_n =$? the RIGHT sum no

and $L_n =$ " " " LEFT " .

$L_n \le \int_{-2}^{0} f \le R_n$

$R_n \le \int_{0}^{6} f \le L_n$

overall, $R_n \le L_n$ on $[-2,6]$, by symmetry

L_n is easy to obtain from R_n, by adding one term & subtracting one term.

$L_n = R_n + \frac{8}{n} f(-2) - \frac{8}{n} f(6)$

$|L_n - R_n| = L_n - R_n = \frac{8}{n}\left(f(-2) - f(6)\right)$

Actual: 1.768308316

$R_n \le \int \le L_n$

$R_n - R_n = 0 \le \int - R_n \le L_n - R_n$

$= \frac{8}{n}\left[e^{-4} - e^{-36}\right] \overset{\text{WANT}}{\le} .00005$ for 4-place accuracy.

$\Rightarrow \frac{8(e^{-4} - e^{-36})}{n} \le (5)(10^{-5})$

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3.3

$$\Rightarrow n \geq \frac{8(e^{-4} - e^{-6})}{(5)(10^{-5})} \approx 2930.502222$$

$\Rightarrow$ Choose $n = 2931$

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B1 5pts

$\frac{1}{2}\sqrt{x^3} > \sqrt{x}$, eventually ($\forall x > 2$)

$\Rightarrow -\frac{1}{2}\sqrt{x^3} < -\sqrt{x}$

$\Rightarrow \sqrt{x^3} - \frac{1}{2}\sqrt{x^3} = \frac{1}{2}\sqrt{x^3} < \sqrt{x^3} - \sqrt{x}$

$\Rightarrow \dfrac{1}{\frac{1}{2}\sqrt{x^3}} > \dfrac{1}{\sqrt{x^3} - \sqrt{x}} \quad \forall x > 2$ ✓

$\int_2^\infty \frac{2}{x^{\frac{3}{2}}}\,dx$ converges ■

$\frac{1}{2}x^{\frac{3}{2}} = x^{\frac{1}{2}}$

$\frac{1}{2}x^{\frac{3}{2}} - x^{\frac{1}{2}} = 0$

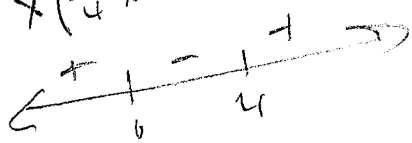
$x^{\frac{1}{2}}\left(\frac{1}{2}x - 1\right) = 0$

B2

$\frac{1}{2}x > \sqrt{x}$

$\frac{1}{4}x^2 - x > 0$

$x\left(\frac{1}{4}x - 1\right) > 0 \quad \forall x > 4$



$\frac{3}{2}x = x + \frac{1}{2}x > x + \sqrt{x}$

$\dfrac{1}{\frac{3}{2}x} < \dfrac{1}{x + \sqrt{x}}$ ✓

$\frac{2}{3}\int_4^\infty \frac{dx}{x}$ diverges ■