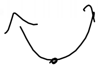
Test 1 Video

https://www.harryzaims.com/202/videos/chapter-06/test-1/

Old Tests

https://www.harryzaims.com/202/old-tests/

1. (10 pts) The function $f(x)=x^2-6x-11$ is 1-to-1 on the restricted domain $[3,\infty)$. Find the inverse function. State its domain and range.



$x^2-6x-11$

$= x^2-6x+3^2-9-11$ $\quad (3,-20)$

$= (x-3)^2-20 \quad (h,k)=(3,-20)$

Restrict: $\mathcal{D}=[3,\infty) = \mathcal{R}(f^{-1})$

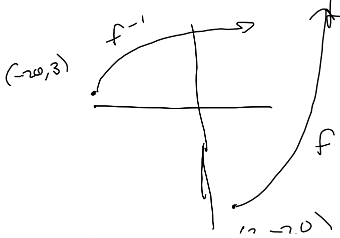
$\mathcal{R}=[-20,\infty) = \mathcal{D}(f^{-1})$

$(x-3)^2-20=y$

$(x-3)^2=y+20$

$x-3=\pm\sqrt{y+20}$ Take top $\frac{1}{2}$, b/c $\mathcal{D}=[3,\infty)$ is RIGHT half,

$x=3+\sqrt{y+20} \rightarrow \boxed{f^{-1}(x)=3+\sqrt{x+20}}$



2. Find $(f^{-1})'(5)$ for $f(x)=x^2-6x-11 \quad (x\ge 3)$, in two ways:

a. (5 pts) Directly, using your answer from #1.

b. (5 pts) Using our theorem for derivative of the inverse..

Didn't use, but $(f^{-1})'(5)$ is awkward...

(a) $f^{-1}(x)=g(x)=3+\sqrt{x+20}$

$(f^{-1})'(5)=g'(5)=\frac{1}{2}(x+20)^{-\frac{1}{2}}\Big|_{x=5} = \frac{1}{2}(25)^{-\frac{1}{2}} = \boxed{\frac{1}{10}=(f^{-1})'(5)}$

(b) $\dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{2(f^{-1}(5))-6}$

$f'(x)=2x-6$ $\quad$ Need $f^{-1}(5)$: Solve $f(x)=5$

$x^2-6x-11=5$

$x^2-6x+3^2=9+16=25$

$(x-3)^2=25$

$x=3\pm\sqrt{25}=3\pm 5$ $\nearrow 8$, $\searrow -2 \notin \mathcal{D}$

$x=8=f^{-1}(5)$

So $(f^{-1})'(5)=\dfrac{1}{2(8)-6}=\dfrac{1}{16-6}$ $\boxed{=\frac{1}{10} \text{ Sweet!} = (f^{-1})'(5)}$

3. (5 pts each) Find the derivative with respect to x. Do not simplify.

a. $y = 2 \cdot 3^{2x^2-3x}$

b. $y = \ln\left(\frac{(x^2-2x)^3}{(2x+1)^5}\right)$ (Hint: Break it up into simpler logs!)

c. $y = \log_3(x^2-2x)$

d. $y = (x^2-3x)^{2x^2+5x}$

e. $y = x^2 \sin^{-1}(x^2-3x)$

f. $y = x^2 \tanh^{-1}(x^2-3x)$

a) $y' = (2\ln 3)(3^{2x^2-3x})(4x-3)$

when in doubt: $\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$

& $b^x = e^{\ln(b^x)} = e^{(\ln b)x} \Rightarrow$

$\frac{d}{dx} b^x = \ln b \cdot b^x$

I always forgot if it was $\ln b$ or $\frac{1}{\ln b}$, so I'd work it out as needed using props of logs & exponents.

b. $y = \ln\left(\frac{(x^2-2x)^3}{(2x+1)^5}\right) = 3\ln(x^2-2x) - 5\ln(2x+1)$

$$\boxed{\Rightarrow y' = 3\left(\frac{2x-2}{x^2-2x}\right) - 5\left(\frac{2}{2x+1}\right)}$$

other ways:

$y = 3\ln x + 3\ln(x-2) - 5\ln(2x+1) \longrightarrow$

$y' = \frac{3}{x} + \frac{3}{x-2} - \frac{10}{2x+1}$; if you factor $x^2-2x = x(x-2)$

so $3\ln(x^2-2x) = 3\ln x + 3\ln(x-2)$

c. $y = \log_3(x^2 - 2x)$

M1 $y' = \left(\frac{1}{\ln 3}\right)\left(\frac{2x-2}{x^2-2x}\right)$

M2 $\log_3(x^2-2x) = \frac{\ln(x^2-2x)}{\ln 3} = \frac{1}{\ln 3}\left(\ln(x^2-2x)\right)$

$\Rightarrow \boxed{y' = \frac{1}{\ln 3}\left(\frac{2x-2}{x^2-2x}\right)}$

Again, I forget whether it's $\frac{1}{\ln b}$ or $\ln b$ so I can always use 121 skills to keep it straight w/ wasting cheat sheet.

d. $y = (x^2 - 3x)^{2x^2+5x}$

$\Rightarrow \ln y = \ln\left((x^2-3x)^{2x^2+5x}\right)$

$= (2x^2+5x)\ln(x^2-3x) \Rightarrow$

$\frac{y'}{y} = (4x+5)\ln(x^2-3x) + (2x^2+5x)\left(\frac{2x-3}{x^2-3x}\right) \Rightarrow$

$\boxed{y' = \left((4x+5)(\ln(x^2-3x)) + \left(\frac{(2x^2+5x)(2x-3)}{x^2-3x}\right)\right)(x^2-3x)^{2x^2+5x}}$

e. $y = x^2 \sin^{-1}\left(x^2 - 3x\right)$

$u = x^2 - 3x$

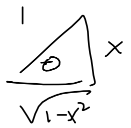
CHAIN

$\Rightarrow y' = (2x)\left(\sin^{-1}(x^2-3x)\right) + (x^2)\left(\sqrt{1-(x^2-3x)^2}\right)(2x-3)$

Keenan's mean.

$\dfrac{1}{\sqrt{1-(x^2-3x)^2}} \cdot (2x-3)$

$\dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{\cos(\sin^{-1} x)} = \dfrac{1}{\sqrt{1-x^2}}$



$f = \sin x$

$f^{-1} = \sin^{-1}(x)$

So

$y' = 2x\left(\sin^{-1}(x^2-3x)\right) + (x^2)\left(\dfrac{1}{\sqrt{1-(x^2-3x)^2}}\right)(2x-3)$

f. $y = x^2 \tanh^{-1}(x^2 - 3x)$

$$\frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d \tanh x}{dx} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$\frac{\frac{e^{2x} + 2 + e^{-2x} - \left[e^{2x} - 2 + e^{-2x}\right]}{4}}{\cosh^2 x} = \frac{\frac{4}{4}}{\cosh^2 x} = \operatorname{sech}^2 x$$

So, $\frac{d}{dx}\left[\tanh^{-1}(x)\right] = \frac{1}{\operatorname{sech}^2\left(\tanh^{-1}(x)\right)}$

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$

This or its equivalent is what you'd need to re-invent this particular wheel. I advise just having the doggone derivatives of inverse hyperbolic trig functions on your cheatsheet.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}$$

Definitely messy enough, you don't want to have to derive!

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\int x e^{x^2-3x}\,dx \qquad u = e^{x^2-3x} \qquad dv = x$$
$$du = (2x-3)e^{x^2-3x}\,dx \qquad v = \frac{x^2}{2}$$

Ugh. Poorly posed question.

$$uv - \int v\,du = \tfrac{1}{2}x^2 e^{x^2-3x} - \tfrac{1}{2}\int x^2 (2x-3)e^{x^2-3x}\,dx$$

b. $\int \frac{dx}{x\sqrt{9-x^2}}$

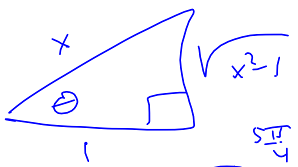
$$\sqrt{9-x^2} = \sqrt{9\left(1-\frac{x^2}{9}\right)} = 3\sqrt{1-\frac{x^2}{3^2}}$$
$$= 3\sqrt{1-u^2}, \text{ when } u = \frac{x}{3} \Rightarrow$$
$$du = \tfrac{1}{3}dx \Rightarrow dx = 3du$$
$$\& \ x = 3u$$

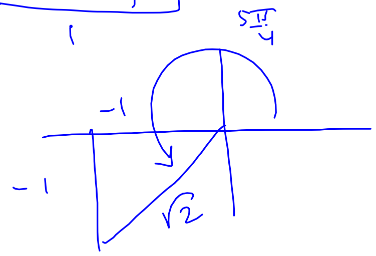
$$= \int \frac{3du}{(3u)(3\sqrt{1-u^2})}$$

$$= \frac{1}{3}\int \frac{du}{u\sqrt{1-u^2}} = \frac{1}{3}\sec^{-1}u + C \ \boxed{= \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C}$$

5. (5 pts each) Simplify:

a. $\tan\left(\sec^{-1}(x)\right) = \tan\theta = \sqrt{x^2-1}$



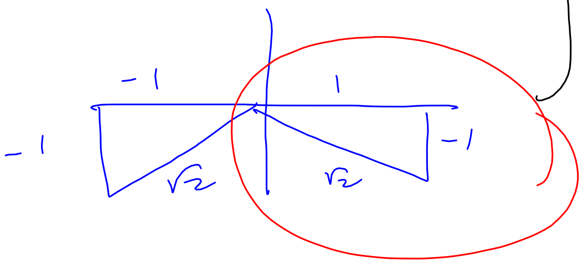


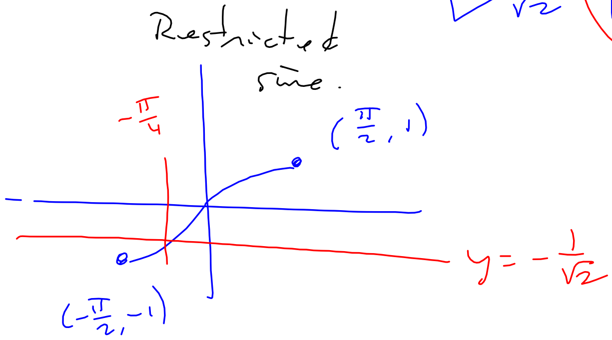
b. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

$\sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \frac{5\pi}{4}$

$= \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

$\sin\theta = -\frac{1}{\sqrt{2}}$





6. The half-life of Carbon-14 is about 5730 years. How old is a fire pit in which 30% of the original Carbon-14 remains?

$A(t)$ = Amt of C_{14} in the sample. (in g?)
t = time in years

$A(t) = A_0 e^{kt}$, where A_0 = initial amt

$\frac{1}{2}$-life is 5730 yrs means

$$A_0 e^{5730k} = \frac{1}{2} A_0$$

$$e^{5730k} = \frac{1}{2}$$

$$5730k = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\boxed{k = \frac{-\ln 2}{5730}}$$

ugh!
$-.00003...-$
Too Digital, dude.

So how old, if 30% of C_{14} remains?

$$A_0 e^{kt} = 0.3 A_0$$

$$e^{kt} = .3$$

$$kt = \ln(.3)$$

$$t = \frac{\ln(.3)}{k} = -\frac{\ln(.3)}{\ln 2} 5730$$

k: $-\frac{\ln 2}{5730}$

$\approx$ 9953 yrs

7. Evaluate $\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 3x}$ in two ways:

a. Factor, cancel, pass to the limit.

b. L'Hopital's rule

(a) $f(x) = \frac{x^2-x-6}{x^2-3x} = \frac{(x-3)(x+2)}{x(x-3)} = \frac{x+2}{x} \xrightarrow{x\to 3} \frac{3+2}{3} = \boxed{\frac{5}{3} = \lim_{x\to 3} f(x)}$

(b) $\lim_{x\to 3} \frac{x^2-x-6}{x^2-3x}$ $\frac{0}{0}$ $\overset{L'H}{=} \lim_{x\to 3} \frac{2x-1}{2x-3} = \boxed{\frac{5}{3} = \lim_{x\to 3} f(x)}$