MAT 202 110 pts
100 Points

Test 4, Spring 2018
Chapter 10

Name KEY

1. (10 pts) Consider the curve given by the parametric equations $x = t^2 - 2t,\ y = t + 3$. Sketch the curve by eliminating the parameter and. Plot enough points (say $t = 0,1,2,3$) to give you a direction of increasing *t*. Indicate the direction of increasing *t* with arrows.

2. (10 pts) Write the integral that gives the area under the parametric curve given by the equations $x = 3\cos(t), y = 3\sin(t)$, from $t = \frac{\pi}{6}$ to $t = \frac{5\pi}{6}$.

Bonus (5 pts) Evaluate the integral from #2.*

3. (10 pts) Show that the surface area of a sphere of radius $r = 3$ is $4\pi r^2 = 4\pi(3)^2 = 36\pi$ by rotating the parametric curve given by the equations $x = 3\cos(t), y = 3\sin(t), 0 \le t \le \pi$ about the *x*-axis. Full credit for writing the integral.

Bonus (5 pts) Evaluate the integral.*

4. Consider the parametric curve defined ty the parametric equations $x = t^2,\ y = t^3 - 18t$
   a. (10 pts) Find an equation of the tangent line to the parametric curve at $t = 3$.

   b. (10 pts) Eliminate the parameter and graph the curve.

   c. (10 pts) Find all points at which the tangent to the curve is horizontal. Label on graph.

   d. (10 pts) Find all points at which the tangent to the curve is vertical. Label on graph.

   e. **Bonus** (5 pts) Find all *t*-intervals on which the curve is concave down.*

5. We analyze the polar curve $r = 1 + 2\cos(2\theta)$.

   a. (10 pts) Check for symmetry.

   b. (10 pts) Sketch $r = 1 + 2\cos(2\theta)$ in *Cartesian* coordinates.

   c. (10 pts) Solve $\frac{dr}{d\theta} = 0$. This answer should be confirmed by the highs and lows of your Cartesian graph.

   d. (10 pts) Write the integral for the area bounded by the curve.

Bonus: (5 pts) Evaluate the integral for part 5d.*

202 TEST 4 Chapter 10 Spring, 2018

① $x = t^2 - 2t$, $y = t + 3$

t	x	y
0	0	3
1	-1	4
2	0	5
3	3	6

$t^2 - 2t = x$

$t^2 - 2t + 1^2 = x + 1$

$(t-1)^2 = x + 1$

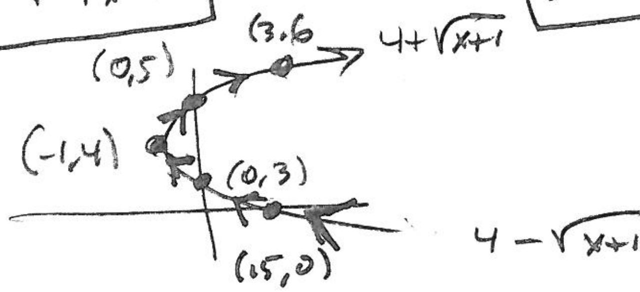
$t - 1 = \pm\sqrt{x+1}$

$t = 1 \pm \sqrt{x+1}$

10pts

$y = 1 + \sqrt{x+1} + 3$ OR $y = 1 - \sqrt{x+1} + 3$

$\boxed{y = 4 + \sqrt{x+1}}$ $\boxed{y = 4 - \sqrt{x+1}}$



$16 = x + 1$

$15 = x$

② $x = 3\cos(t)$, $y = 3\sin(t)$ $\frac{\pi}{6} \le t \le \frac{5\pi}{6}$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} y\,dx = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 3\sin(t)\,(-3\sin(t)\,dt)$$

$dx = -3\sin(t)\,dt$

$$\boxed{-9\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2(t)\,dt}$$

10pts

is wrong sign, but $t = \frac{\pi}{6}$ to $t = \frac{5\pi}{6}$ takes x from pos to neg.

202

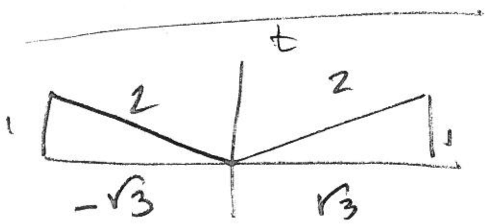
BONUS AFTER #2 $\quad \sin^2(t) = \frac{1-\cos 2\theta}{2}$

$$-9\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1-\cos(2t)}{2}\,dt =$$

$$-9\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\,dt + 9\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\cos(2t)\,dt$$

$$= -\frac{9}{2}t\Big]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{9}{2}\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos(2t)\cdot 2\,dt$$

$$= -\frac{9}{2}\left[\frac{5\pi-\pi}{6}\right] + \frac{9}{4}\Big[\sin(2t)\Big]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$



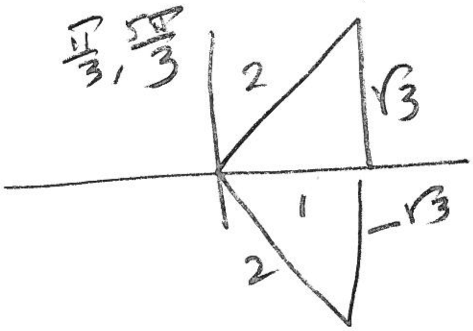
$$= -\frac{9}{2}\cdot\frac{4\pi}{6} + \frac{9}{4}\left[-\frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{2}\right)\right]$$

$2t = \frac{\pi}{6}, \frac{5\pi}{6}$

$$= -3\pi + \frac{9}{4}\left[-\sqrt{3}\right] = \boxed{-3\pi - \frac{9\sqrt{3}}{4}}$$

5pts

$\frac{\pi}{3}, \frac{5\pi}{3}$



202 T4

3

$$2\pi \int_0^\pi y\, ds = 2\pi \int_0^\pi 3\sin(t)$$

$x = 3\cos(t)$
$y = 3\sin(t)$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\left(\frac{dx}{dt} = -3\sin(t),\ \frac{dy}{dt} = 3\cos t\right)$$

$$\Rightarrow 2\pi \int_0^\pi 3\sin(t)\sqrt{9(\sin^2(t)+\cos^2(t))}\, dt$$

$$= \boxed{18\pi \int_0^\pi \sin(t)\, dt =}$$

10pts OR some minor variation on

BONUS

$$= 18\pi \left[-\cos(t)\right]_0^\pi = 18\pi\left[-(-1) - (-1)\right]$$

$$\boxed{= 36\pi}$$

5 Bonus

202 T3

(4) $x=t^2$, $y=t^3-18t$, $t=3$. Tangent Line

(a) $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-18}{2t}=y'(t) \longrightarrow$$

$$y'(3)=\frac{3(3^2)-18}{2(3)}=\frac{27-18}{6}=\frac{9}{6}=\boxed{\frac{3}{2}=m_{tan}}$$

$x(3)=9=x_1$

$y(3)=27-54=-27=y_1$

$y=m(x-x_1)+y_1$ 10pts

$$\boxed{y=\frac{3}{2}(x-9)-27}$$

(b) $$t=\pm\sqrt{x} \longrightarrow y=\begin{cases}(\sqrt{x})^3-18\sqrt{x}\\ \text{OR}\\ (-\sqrt{x})^3-18(-\sqrt{x})\end{cases}$$

$$=\begin{cases}x^{\frac{3}{2}}-18x^{\frac{1}{2}}\\ \text{OR}\\ 18x^{\frac{1}{2}}-x^{\frac{3}{2}}\end{cases}=\begin{cases}x^{\frac{1}{2}}(x-18)\\ \text{OR}\\ x^{\frac{1}{2}}(18-x)\end{cases}$$

$\frac{9}{\frac{3}{2}}=\frac{18}{3}=6$

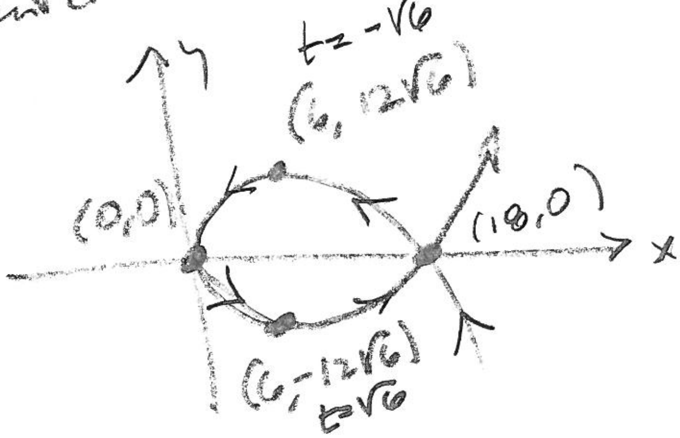
$$\frac{dy}{dx}=\begin{cases}\frac{3}{2}x^{\frac{1}{2}}-9x^{-\frac{1}{2}}\\ 9x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}\end{cases} \underline{\underline{\text{SET}}}\ 0 \longrightarrow \begin{cases}\frac{3}{2}x^{-\frac{1}{2}}(x-6)=0\\ \frac{3}{2}x^{-\frac{1}{2}}(6-x)=0\end{cases}$$

$x=6$ $y'=0$

$x=0$ $y'\nexists$

202 TEST 4. 1st 6

4b cnt'd



Horiz. Tan. Labels c 10pts
Vertical Tan. Labels d 10pts

$\frac{2}{2}\cdot\frac{4}{3} = 2\cdot 3 = 6$

$y=0$ @ $x=18,\ x=0$

$y(6) = \pm(\sqrt{6}^3 - 18\sqrt{6}) = \pm(-12\sqrt{6})$

$y'=0$ @ $x=6$

$y'' = \frac{3}{4}x^{-\frac{1}{2}} + \frac{9}{2}x^{-\frac{3}{2}} = \frac{3}{4}x^{-\frac{3}{2}}(x+6) \neq 0$

$\frac{x^{-\frac{1}{2}}}{x^{-\frac{3}{2}}} = x^{\frac{3}{2}-\frac{1}{2}} = x$ ✓

$\Longrightarrow x=-6$ @ so never

$3t^2-18=0 \Rightarrow t^2=6 \Rightarrow t=\pm\sqrt{6}$

$x(\sqrt{6}) = 6$

$y(\sqrt{6}) = \sqrt{6}^3 - 18\sqrt{6} = -12\sqrt{6}$

$= 6\sqrt{6} - 18\sqrt{6}$

Parametric Version

$\frac{dy}{dx} = \frac{3t^2-18}{2t}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{6t(2t)-(3t^2-18)(2)}{(2t)^2}}{2t}$

202

4 cntd

$$= \frac{12t^2 - 6t^2 + 36}{(2t)^3} = \frac{6t^2+36}{8t^3} = \frac{3t^2+18}{4t^3}$$

4e Bonus

$$= \frac{3}{2}\left(\frac{t^2+6}{t^3}\right) > 0 \text{ for } t > 0$$

$$< 0 \text{ for } t < 0$$

$\boxed{\text{Concave down } \forall\, t \in (-\infty, 0)}$

5 $r = 1 + 2\cos(2\theta)$

a Pole : $-r = 1 + 2\cos(2\theta)$ No

YES

$r = 1 + 2\cos(2(\theta+\pi)) = 1 + 2\cos(2\theta + 2\pi)$

$= 1 + 2\cos(2\theta)$ Yes

Polar Axis : $r = 1 + 2\cos(2(-\theta))$

YES

$= 1 + 2\cos(-2\theta)$

$= 1 + 2\cos(2\theta)$

10 pts

202 T3

5a

$x = \frac{\pi}{2}$?

YES

$r = 1 + 2\cos(2(\pi - \theta))$

$= 1 + 2\cos(2\pi - 2\theta)$

$= 1 + 2\cos(-2\theta)$

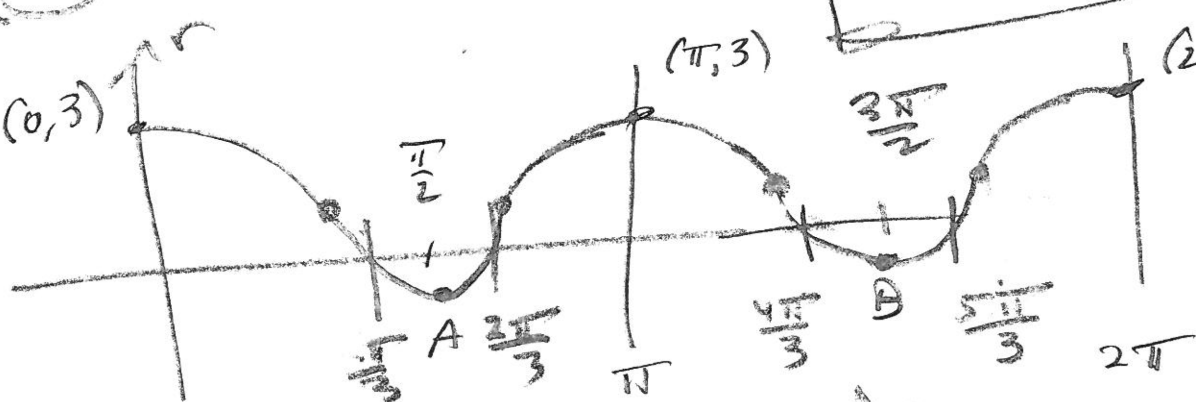
$= 1 + 2\cos(2\theta)$

$A = (\frac{\pi}{2}, -1)$

$B = (\frac{3\pi}{2}, -1)$

loops

5b $r = 1 + 2\cos(2\theta)$



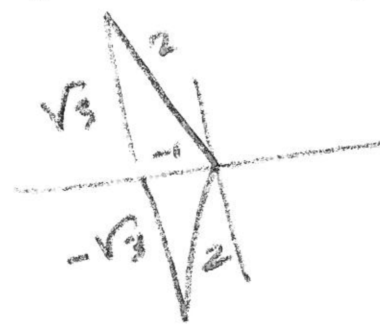
$2\cos(2\theta) + 1 = 0$

$2\cos(2\theta) = -1$

$\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\theta = \frac{2\pi}{6}, \frac{4\pi}{6} = \frac{\pi}{3}, \frac{2\pi}{3}$



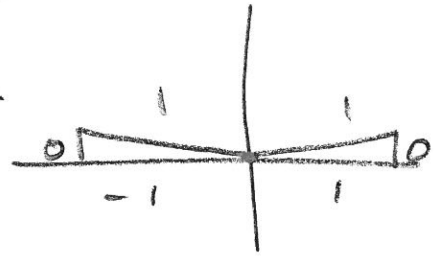
Then it goes another whole period

So $\frac{4\pi}{3}, \frac{5\pi}{3}$ also

202 T3

5c From graph, $\frac{dr}{d\theta} = 0$ @

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



$r = 1 + 2\cos(2\theta)$

$\frac{dr}{d\theta} = -4\sin(2\theta) \overset{\text{set}}{=} 0$

$\Rightarrow 2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\rightarrow \boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi}$

$r(0) = 3$

$r(\frac{\pi}{2}) = -1$

$r(\pi) = 3$

$r(\frac{3\pi}{2}) = -1$

$r(2\pi) = 3$

5d $\boxed{\text{Area} = \frac{1}{2}\int_0^{2\pi} (2\cos(2\theta)+1)^2 \, d\theta}$ 10pts

$= \frac{1}{2}\int_0^{2\pi} (4\cos^2(2\theta) + 4\cos(2\theta) + 1) \, d\theta$ BONUS

$= \frac{1}{2}\int_0^{2\pi} \left(4\left(\frac{1}{2}\right)(\cos(4\theta)+1) + 4\cos(2\theta) + 1\right) d\theta$

202 T3

Bonus cnt'd (After 5d)

$$= \frac{1}{2}\int_0^{2\pi} 2\cos(4\theta)\,d\theta + \frac{1}{2}\int_0^{2\pi} 4\cos(2\theta)\,d\theta + \int_0^{2\pi} 3\,d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \cos(4\theta)\cdot 4\,d\theta + \frac{1}{2}\cdot 2\int_0^{2\pi} \cos(2\theta)\cdot 2\,d\theta$$

$$+ 6\pi$$

$$= \frac{1}{4}\Big[-\sin(4\theta)\Big]_0^{2\pi} + \Big[-\sin(2\theta)\Big]_0^{2\pi} + 6\pi$$

$$= -\frac{1}{4}[0-0] - [0-0] + \boxed{6\pi}$$