Compute the limit

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)$$

Involves taking a log and L'Hopital's Rule.

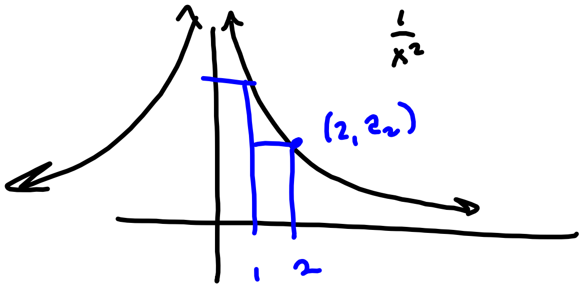
A nice 11.1 bonus question also in the notes.

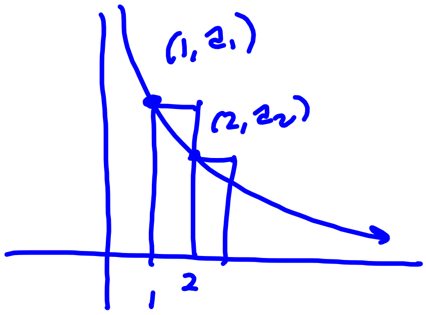
Prove $\lim_{n\to\infty}\frac{n}{n+1}=1$

Section 11.3: I want folks to nail the improper integral skill, and this is our last chance to do so.

We know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the $p = 2$-test, because $2 > 1$. Show that it converges by the integral test.

Hint: $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \sum_{n=2}^{\infty}\frac{1}{n^2}$ converges if and only if $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges. The first few terms always add up to something finite. The question is always about whether or not the infinite number of terms that come after those first few terms.





$$\sum_{n=1}^{\infty} < \int_0^{\infty} \qquad\qquad \sum_{n=1}^{\infty} > \int_1^{\infty}$$

But for sure z,

$$\sum_{n=2}^{\infty} < \int_1^{\infty}$$ & we can evaluate that.

$$\sum_{n=1}^{\infty} = 1 + \sum_{n=2}^{\infty}$$

$$\sum_{n=2}^{\infty} < \int_1^{\infty}\frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t x^{-2}\,dx$$

$$\int_1^t x^{-2}\,dx = -x^{-1}\Big]_1^t = -t^{-1} - (-1^{-1})$$

$$= 1 - \frac{1}{t} \xrightarrow{t\to\infty} 1$$ converges, so $\sum_{n=1}^{\infty}$ converges.

$$\frac{x^{-2+1}}{-2+1} = -x^{-1}$$

Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the Integral Test.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_2^{\infty} x^{-\frac{1}{2}}\,dx$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test.

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$= \int x^{-1}\,dx$$

$$\int_2^{\infty} < \sum_1^{\infty} < \int_1^{\infty}$$

All 3 either converge or all 3 diverge.

otherwise $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

Let $S = \sum_{n=1}^{\infty} 2^{-n}$.

Give a floor and a ceiling on the error $|R_6|$ for the 6th partial sum, S_6.

$$\int_7^{\infty} < R_6 < \int_6^{\infty}$$

$$\sum_{n=7}^{\infty} 2^{-n}$$

$$S = S_6 + R_6$$

$$S_6 + \int_7^{\infty} < \underbrace{S_6 + R_6}_{= S} < S_6 + \int_6^{\infty}$$

Show that $S = \sum_{n=1}^{\infty} \frac{n^2 - 3n - 6}{5n^3 + 4n^2 - 2n + 998{,}462}$ converges by the Limit Comparison Test. → Doh!

$= \sum_{n=1}^{\infty} a_n$ $\qquad \sum_{n=1}^{\infty} \frac{n^2 - 3n - 6}{5n^4 + 4n^2, \text{etc.}}$ <u>This</u> converges.

Let $b_n = \frac{n^2}{5n^4} = \frac{1}{5n^2}$

Then examine $\left| \frac{a_n}{b_n} \right| \xrightarrow{n \to \infty} 1$ = the limit

$$\frac{n^2\left(1 - \frac{3}{n} - \frac{6}{n^2}\right)}{n^4\left(5 + \frac{4}{n} - \frac{2}{n^2} + \frac{998{,}462}{n^3}\right)} \cdot \frac{5n^2}{1}$$

$\frac{6}{n^2} \to 0$

$$= \frac{1}{n^2}\left(\frac{1 - \_\_\_}{5 + \_\_\_}\right) \cdot \frac{5n^2}{1} \quad \underline{n \to \infty}$$

$$\frac{1}{n^2}\left(\frac{1}{5}\right)\left(\frac{5n^2}{1}\right) = 1$$

Show that $S=\sum_{n=1}^{\infty}\frac{n^2-3n}{5n^3+4n^2}$ converges by the Direct Comparison Test.

$3 \to 4$

$$\frac{n^2-3n}{5n^4+4n^2} < \frac{n^2}{5n^2+4n^2} < \frac{n^2}{5n}$$

$$\frac{n^2-3n}{5n^4+4n^2} < \frac{n^2}{5n^4+4n^2} < \frac{n^2}{5n^4} = \frac{1}{5n^2} = b_n$$

$$\sum \frac{1}{5n^2} \quad p=2$$

Bonus Show that $S=\sum_{n=1}^{\infty}\frac{n^2+3n}{5n^3-4n^2}$ converges by the ~~Limit~~ Direct Comparison Test.

$$n^2+3n < n^2+3\left(\frac{n^2}{2}\right) = n^2+\frac{3}{2}n^2 = \frac{5}{2}n^2$$

$$5n^4-4n^2 > \underline{5n^4-4\left(\frac{n^4}{2}\right)}$$

$$\frac{n^2+n}{n^4-n^2} \qquad \frac{n^3+3n}{5n^4-4n^2} < \frac{n^3+3\left(\frac{n^2}{2}\right)}{5n^4-4\left(\frac{n^4}{2}\right)}$$

Haven't stressed term-by-term differentiation or integration.

$$\frac{1}{(x-1)^2} = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right] =$$

$$= -1(1-x)^{-2}(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}$$

So find $\frac{1}{1-x}$ & differentiate:

$$f(x)\ \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\Rightarrow f'(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{k=0}^{\infty} x^k\right] = \sum_{k=0}^{\infty} kx^{k-1}$$

$$= \sum_{k=1}^{\infty} kx^{k-1}$$

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\downarrow \quad 1 \quad 2 \quad 3$$

$$\Rightarrow f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}$$