$$\sum_{n=0}^{\infty} c_n x^n \qquad \text{Power Series}$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

Recall

$$S_n = \sum_{k=0}^{n} ax^k = a\left(\frac{1-x^{n+1}}{1-x}\right)$$

$$\sum_{k=1}^{n} ax^{k-1} = a\left(\frac{1-x^n}{1-x}\right)$$

If $|x| < 1$, then

$$\sum_{k=1}^{\infty} ax^{k-1} = a\left(\frac{1}{1-x}\right)$$

$$x^n \xrightarrow{n \to \infty} \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } x > 1 \\ \text{Nothing} & \text{if } x \le -1 \end{cases}$$

Check this:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

So....

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots$$

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$\frac{3}{9+x^2} = \frac{3}{9(1+\frac{x^2}{9})} = \frac{3}{9}\left(\frac{1}{1-(-\frac{x^2}{9})}\right)$$

$$= \frac{1}{3}\left[1 + \left(-\frac{x^2}{9}\right) + \left(-\frac{x^2}{9}\right)^2 + \left(-\frac{x^2}{9}\right)^3 + \cdots\right]$$

$$= \frac{1}{3}\left[1 - \frac{x^2}{9} + \frac{x^4}{81} - \frac{x^6}{729} + \cdots\right]$$

Let's Get Tricky:
Power Series Representation for

$$\boxed{\frac{1}{(x+1)^2}}$$

$$\int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx = \int u^{-2}\,dx = \frac{u^{-1}}{-1} + C$$

$u = x+1$
$du = dx$

$$= -\frac{1}{x+1} + C$$

$$-\frac{1}{1+x} = -\left[1 - x + x^2 - x^3 + \cdots\right]$$

$$\frac{d}{dx}\left[-(x+1)^{-1}\right] = -\left(-1\,(x+1)^{-2}\right) = \frac{1}{(1+x)^2}$$

$$\stackrel{?}{=} -\left[0 - 1 + 2x - 3x^2 + 4x^3 + \cdots\right]$$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots = \frac{1}{(1+x)^2}$$

using term-by-term differentiation, which is totally legit. So is term-by-term integration.

Recall $\sum \frac{1}{n}$ diverges.

Consider

$$S = \sum_{n=1}^{\infty} \frac{1}{n} x^n.$$ Check it with ratio test.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \left| \frac{n}{n+1} \cdot x \right|$$

$$\xrightarrow{n \to \infty} |x| \overset{\text{WANT}}{<} 1 = R = \text{Radius of convergence for } S.$$

FOR SURE, this converges $\forall\, x \in (-1, 1)$

We're $\underline{\text{NOT}}$ sure of the endpoints $x = \pm 1$.

$x = 1$: $\sum \frac{1}{n}(1)^n = \sum \frac{1}{n}$ Diverges.

$x = -1$: $\sum \frac{1}{n}(-1)^n = \sum (-1)^n \frac{1}{n}$ Converges.

So $\boxed{R = 1}$ & I = Interval of convergence is $\boxed{I = [-1, 1)}$

$\sum c_n x^n$ is centered @ $x=0$

$\sum c_n (x-5)^n$ is centered @ $x=5$

$\sum \frac{1}{n}(x-5)^n$ $\quad R=1, \quad I=[4,6)$

Endpoints: $5-1=4$, $5+1=6$ $\Big\}$ Test, separately in general.

Ratio & Root Tests are for ABSOLUTE CONVERGENCE, only, i.e.,

$\sum |a_n|$ converges.

(12) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\cdot 3^n}$

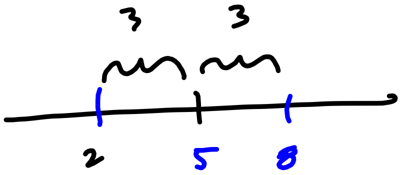
$$\left|\frac{x^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{x^n}\right| = \frac{1}{3}|x| \overset{\text{want}}{<} 1 \Rightarrow |x|<\boxed{3\equiv R}$$

$x=-3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(3)^n}{(n)(3^n)} = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ converges

$$(-3)^n = ((-1)(3))^n = (-1)^n(3)^n$$

$x=-3$: ... $\sum \frac{1}{n}$ diverges.

$$\boxed{I=[-3,3)}$$



$\sum \frac{(x-5)^n}{n\cdot 3^n}$

... $|x-5|<3$

$-3 < x-5 < 3$

$2 < x < 8$

$$\frac{x^2}{(1+x)^2} = x^2\underbrace{\left[\frac{1}{(1+x)^2}\right]}$$

Do this. Then multiply term-by-term by the x^2

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{d}{dx}\left[\ \ \right] = -\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \cdots$$

So $$\frac{x^2}{1+x^2} = x^2 - 2x^3 + 3x^4 - 4x^5 + \cdots$$