§ 11.2

$\sum_{k=1}^{\infty} a_k$ The question is "Does it converge?"

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \text{Geometric}$$

$$\sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} = S_n = n^{\text{th}} \text{ partial sum}$$

$a = R$
$r = (1+i)$

$$= a\left(\frac{1-r^n}{1-r}\right) \xrightarrow{n \to \infty} a\left(\frac{1}{1-r}\right) \text{ if } -1 < r < 1$$

If $|r| \geq 1$, then Diverges.

Examples: Annuities, Loans,

$$R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1}$$

i = interest rate per period $= \frac{r}{m}$, where

r = annual interest rate

m = # of periods per year. R = periodic pmt

t = " " years

$n = mt$ = # periods.

Assume pmts are at the end of each period.

$$S_n = \sum_{k=0}^{n-1} R(1+i)^k = \sum_{k=1}^{n} R(1+i)^{k-1}$$

$$= R\left(\frac{1-(1+i)^n}{1-(1+i)}\right) = R\left(\frac{1-(1+i)^n}{-i}\right)$$

$$= R\left(\frac{(1+i)^n - 1}{i}\right) = \text{Future value of an annuity}$$

$$\xrightarrow{n \to \infty} \infty$$

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^k \text{ converges for } -1 < r < 1$$

p-series

$$S = \sum_{k=1}^{\infty} \frac{1}{n^p}$$

Converges if $p > 1$

Diverges if $p \le 1$

$p=1$ is Harmonic Series

$\sum_{k=1}^{\infty} \frac{1}{n}$ diverges

$$S = a_1 + a_2 + \dots + a_{n-1} + a_n + a_{n+1} + \dots$$

$= S_n + R_n =$ n^{th} partial sum + n-tail

$= (a_1 + \dots + a_n) + (a_{n+1} + \dots \quad)$

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§11.3 Integral Test.

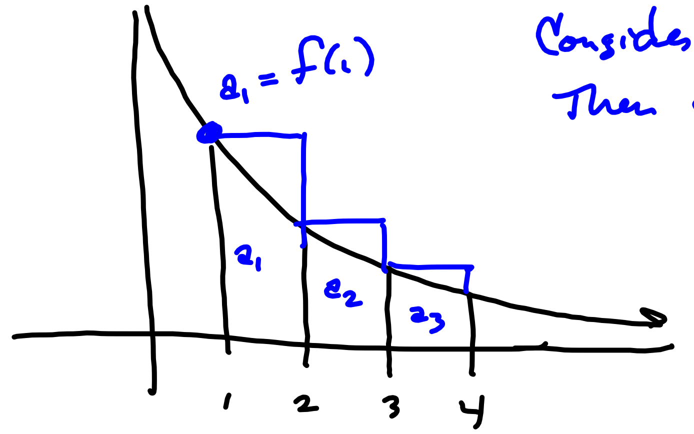
a_n -vs- $f(n) = f(x)$

$\sum_{k=1}^{\infty} a_k$ -vs- $\int_1^{\infty} f(x)\,dx$

Both converge or both diverge.

Assume $f(x) > 0$ & $f(x)$ is decreasing

Then f is continuous & $f(x) \xrightarrow{x \to \infty} 0$



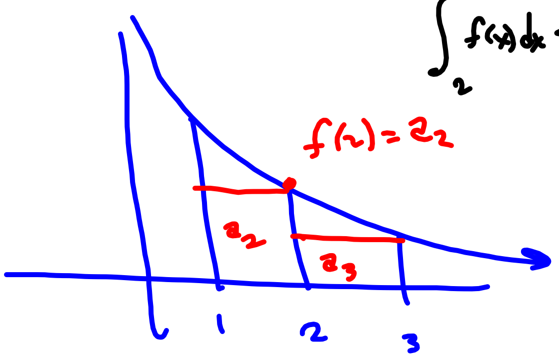
Consider $a_n = f(n)$

Then consider $\sum_{k=1}^{\infty} a_k$

So, $\sum_{k=1}^{\infty} a_k \geq \int_1^{\infty} f(x)\,dx$

But $\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx$

$$\int_2^{\infty} f(x)\,dx \leq \sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k$$



If $f(x)$ is continuous then $\int_1^n f(x)\,dx \ \exists$

Hopefully, $\sum_{k=1}^{10} a_k \approx \sum_{k=1}^{\infty} a_k$, i.e.,

hopefully, $\sum_{k=11}^{\infty} a_k$ is small.

$$R_{10} = \sum_{k=11}^{\infty} a_k = S - S_{10}$$

$$S = S_n + R_n = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k$$

$$\sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} f(x)\,dx \le \sum_{k=n}^{\infty} a_k$$

Gives us a bound on the error!

$$|R_n| = |S - S_n|$$

How many terms are needed to come within $\frac{1}{100}$ of $\sum_{k=1}^{\infty} \frac{1}{n^2}$?

$$|R_n| = \left|\sum_{k=n+1}^{\infty} a_k\right| \le \left|\int_n^{\infty} \frac{1}{x^2}\,dx\right| = \lim_{m\to\infty} \int_n^m x^{-2}\,dx$$

$$= \lim_{m\to\infty} \left[-1x^{-1}\right]_n^m = \lim_{m\to\infty}\left(-\frac{1}{m} - \left(-\frac{1}{n}\right)\right) = \frac{1}{n} < \frac{1}{100}$$

$\rightarrow$ $\boxed{100 < n}$

If $n > 100$, then

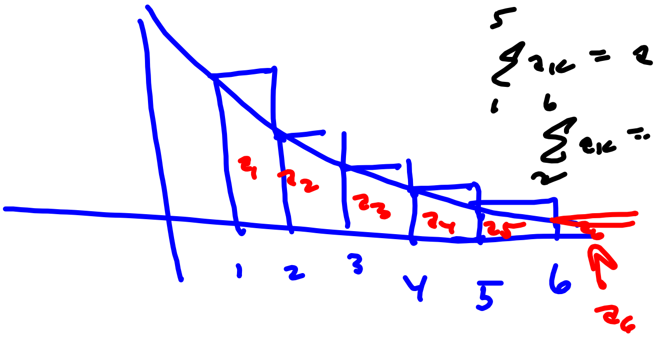
$$|S - S_n| = |R_n| < \frac{1}{100}$$

Old § 11.3 #2

f cnt's, $f > 0$, $f' < 0$, $\forall\, x \geq 1$

$a_n \equiv f(n)$

$$\int_1^6 f(x)\,dx < \sum_{k=1}^{5} a_k > \sum_{k=2}^{6}$$



$$\sum_{1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$$

$$\sum_{2}^{6} a_k = a_2 + a_3 + a_4 + a_5 + a_6$$

Think of

$$\frac{1}{n^2}, \quad \frac{\sqrt{n} - 7}{n^{\frac{5}{2}} + n}, \quad \text{and}$$

anything like this as $\frac{1}{n^2}$