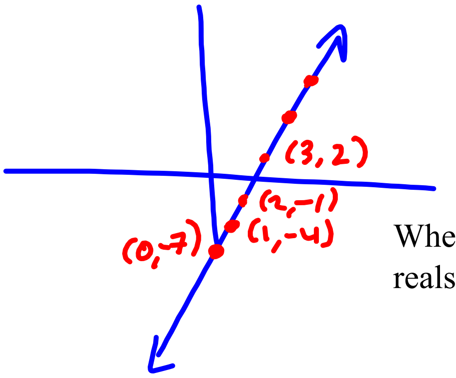
Recall function $f: \mathbb{R} \longrightarrow \mathbb{R}$

like $f(x) = 3x - 7$

Sequences: $f: \mathbb{N} \longrightarrow \mathbb{R}$

$f(n) = 3n - 7$



Red dots are a graph of the sequence $f(n)$

When in doubt about f(n), graph f(x) as a function from the reals into the reals.

We are most interested in what happens as n goes to infinity.

$$\lim_{n \to \infty} f(n) = \lim_{x \to \infty} f(x) = \boxed{\infty}$$

Special case of "divergence."

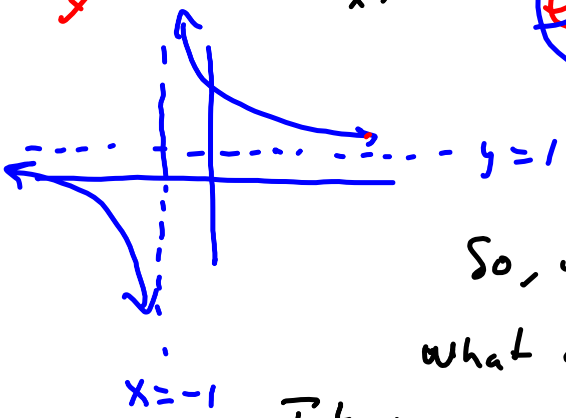
Recall limits:

$f(x) = \frac{x}{x+1}$

$\lim_{x\to\infty} \frac{x}{x+1} =$

$$\frac{2x^5 + 19x^3 - 592872x}{7x^5 - 1} \xrightarrow{x\to\infty} \frac{2}{7}$$

$$\frac{x}{x+1} = \frac{x}{x(1+\frac{1}{x})} = \frac{1}{1+\underbrace{\frac{1}{\underset{\to\infty}{x}}}_{\to 0}} \xrightarrow{x\to\infty} 1$$



So, intuitively, $\lim_{x\to\infty} f(x) = 1$

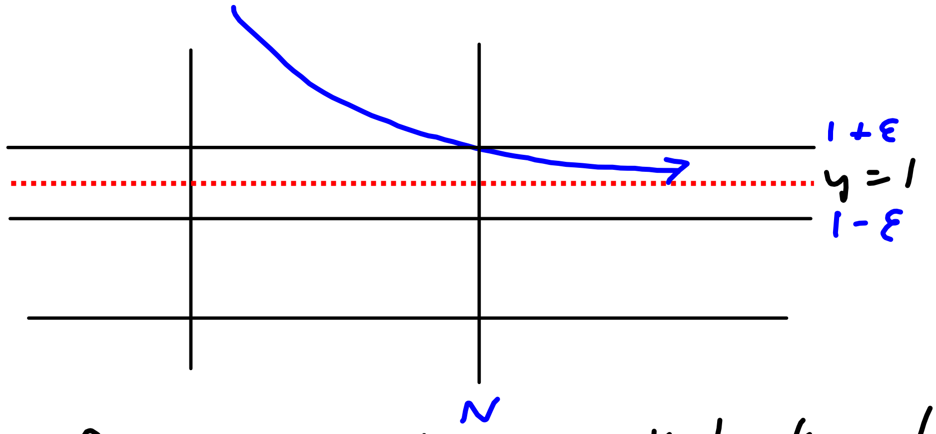
What does this mean?

It means we can make f(x) arbitrarily close to 1

$|f(x) - 1| < \varepsilon$ for arbitrarily small $\varepsilon > 0$, eventually (for BIG-ENOUGH x.).

FORMALLY $\boxed{\lim_{x\to\infty} f(x) = 1}$ means

for any $\varepsilon > 0$, I can find an $N > 0$ such that any time $x > N$, we have $|f(x) - 1| < \varepsilon$.



To Prove this is true, work backwards.

Want $|f(x)-1| < \varepsilon \quad \forall\ x > N$

$$\left|\frac{x}{x+1} - 1\right| < \varepsilon$$

$$\left|\frac{x-(x+1)}{x+1}\right| = \left|\frac{x-x-1}{x+1}\right| = \left|\frac{-1}{x+1}\right| = \left|\frac{1}{x+1}\right|$$

Since $x >$ BIG $= \frac{1}{x+1} < \varepsilon$

$$\Rightarrow \quad 1 < \varepsilon(x+1)$$

$$\varepsilon(x+1) > 1$$

$$x+1 > \frac{1}{\varepsilon}$$

$$x > \frac{1}{\varepsilon} - 1 = \frac{1-\varepsilon}{\varepsilon}$$

wanna keep this positive, so assume $0 < \varepsilon < 1$

DEFINE $N = \frac{1-\varepsilon}{\varepsilon}$

Proof

Let $\varepsilon > 0$ be given. Assume $\varepsilon < 1$. Define $N = \frac{1-\varepsilon}{\varepsilon}$. Then, if $x > N$, we have

$$\left| f(x) - 1 \right| = \left| \frac{x}{x+1} - 1 \right| = \left| \frac{x-(x+1)}{x+1} \right|$$

$$= \left| \frac{x-x-1}{x+1} \right| = \left| \frac{-1}{x+1} \right| = \frac{1}{x+1} < \frac{1}{N+1}$$

$$= \frac{1}{\frac{1-\varepsilon}{\varepsilon}+1} = \frac{1}{\frac{1-\varepsilon+\varepsilon}{\varepsilon}} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \quad \blacksquare$$

For sequences, we say

$f(n) = a_n$ = n^{th} term of the sequence

Sequence Notation $\{a_n\}_{n=1}^{\infty} = \{a_n\}_{n \in \mathbb{N}}$

$= a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$

$\mathbb{N} = \{1, 2, \ldots\}$

Claim: $\lim_{n\to\infty} \frac{1}{n} = 0$

Scratch

want $\left|\frac{1}{n} - 0\right| < \varepsilon$

$\frac{1}{n} < \varepsilon$

$1 < \varepsilon n$

$N \equiv \frac{1}{\varepsilon} < n$

Proof Let $\varepsilon > 0$ be given. Define $N = \frac{1}{\varepsilon}$.

Then $\forall$ $n > N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \; \blacksquare$$

$\frac{3}{4}, \frac{7}{9}, \frac{11}{16}, \ldots$

Adding 4 : $4n$

$n=1$ ~~$\frac{n+2}{n+3} = \frac{3}{4}$~~

$n=2$ ~~$\frac{n+2}{n+3} = \frac{4}{5}$~~ Nah

$n=1$, I need it to be a 3

$4n: 4, 8, 12, 16$

If we started $n=0$, then

$4n+3 = 4(\overset{n-1}{0})+3 = 3$ $n=1$

$4(\overset{n-1}{1})+3=7$ $n=2$

$4(\overset{n-1}{2})+3 = 11$ $n=3$

$4(n-1)+3 = 4n-4+3 = 4n-1$

$n=1: \ 4 = 2^2 = (n+1)^2$

$n=2: \ 9 = 3^2 = (n+1)^2$

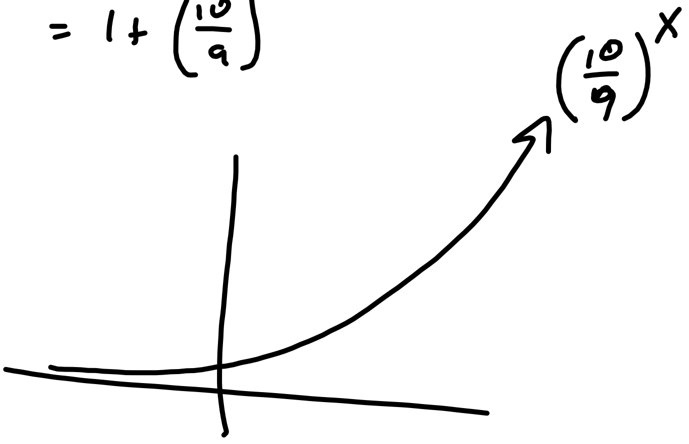
$$\frac{4(n-1)+3}{(n+1)^2} \quad \text{OR} \quad \frac{4n-1}{(n+1)^2}$$

Sequence Bounded Above
& Increasing Converges.
Monotone Bounded (Dominated)
convergence
theorem.

$$a_n = \frac{3n}{6n+1} \xrightarrow{n\to\infty} \frac{3}{6} = \frac{1}{2}$$

$$a_n = 1 + \frac{10^n}{9^n} \xrightarrow{n\to\infty} \infty$$

$$= 1 + \left(\frac{10}{9}\right)^n$$



$$a_n = e^{\frac{1}{n}} \xrightarrow{n\to\infty} e^0 = 1$$

$$\ln(a_n) = \ln\left(e^{\frac{1}{n}}\right) = \frac{1}{n}\ln e = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

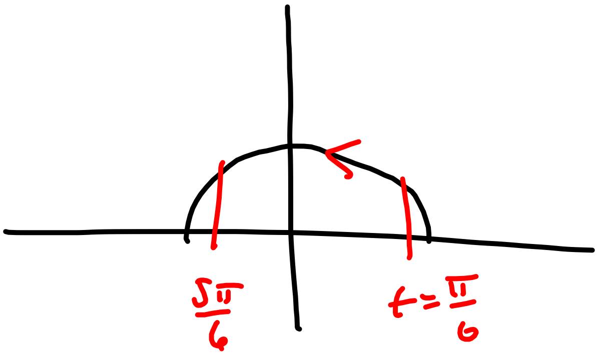
$$\Rightarrow \lim_{n\to\infty} e^{\ln(a_n)} = e^0 = 1$$

$$a_n = \frac{n^2}{e^n} \xrightarrow{n\to\infty} 0 \quad \left(\frac{\infty}{\infty}\right)$$

$$\frac{n^2}{e^n} \xrightarrow[L'H]{n\to\infty} \underset{\frac{\infty}{\infty}}{\frac{2n}{e^n}} \xrightarrow[L'H]{n\to\infty} \frac{2}{\underbrace{e^n}_{\to\infty}} \xrightarrow{n\to\infty} 0.$$

$$a_n = \left(1+\frac{2}{n}\right)^n \xrightarrow{n\to\infty} e^2$$

$$\ln(a_n) = \underset{\infty\cdot 0}{n\ln\left(1+\frac{2}{n}\right)} = \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \frac{0}{0} \text{ etc}$$



$$-\int_{\pi/6}^{5\pi/6} \overset{3\sin t}{y}\, \underset{-3\sin t\, dt}{dx}$$

$\int y\, dx$ Area

$$-\int_{\pi/6}^{5\pi/6} 9\sin^2 t\, dt$$

$\int ds$ arc length

$x = 3\cos t$

$dx = -3\sin t\, dt$