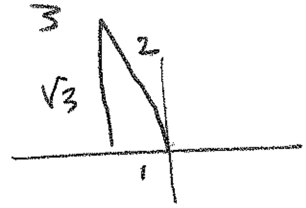
202 §11.2 #s 4, 8, 17, 18, 22, 26, 32, 35*

#s 1–14 Find tan to curve @ point defined by t-value. Also find $\frac{d^2y}{dx^2}$ @ that point.

(4) $x = \cos t$, $y = \sqrt{3}\cos t$ @ $t = \frac{2\pi}{3}$



$\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sqrt{3}\sin t$

$t = \frac{2\pi}{3} \Rightarrow$

$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, $y = \sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$(x_0, y_0) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$, regardless of t!

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = 0}$!? looking at a straight line?

tan line: $\boxed{y = \sqrt{3}\left(x + \frac{1}{2}\right) - \frac{\sqrt{3}}{2}}$

$= \sqrt{3}x + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= \sqrt{3}x$ is what bet the curve looks like!

$t = \arccos(x)$

$y = \sqrt{3}\cos(\arccos x)$

$= \sqrt{3}x$! :)

202 § 11.2 #s 8, 17, 18, 22, 26, 32, 35

(8) $x = -\sqrt{t+1} \qquad y = \sqrt{3t} \quad , \; t = 3$

$x(3) = -2 \qquad y(3) = 3 \rightsquigarrow (-2, 3) = (x_0, y_0)$

$\frac{dx}{dt} = -\frac{1}{2}(t+1)^{-\frac{1}{2}} \qquad \frac{dy}{dt} = \frac{1}{2}(3t)^{-\frac{1}{2}}(3)$

$\frac{dy}{dx} = \frac{(3/2)(3t)^{-\frac{1}{2}}}{-\frac{1}{2}(t+1)^{-\frac{1}{2}}} = \frac{-3\sqrt{t+1}}{\sqrt{3t}} = -3\left(\frac{t+1}{3t}\right)^{\frac{1}{2}} = \frac{dy}{dx}$

$\Rightarrow m_{tan} = -3\left(\frac{3+1}{3(3)}\right)^{\frac{1}{2}} = -3\left(\sqrt{\frac{4}{9}}\right) = -3\left(\frac{2}{3}\right) = -2 = m$

$\boxed{y = -2(x+2) + 3}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{(-3/2)\left(\frac{t+1}{3t}\right)^{-\frac{1}{2}}\left(\frac{3t-(t+1)(3)}{9t^2}\right)}{-\frac{1}{2}(t+1)^{-1/2}}$

$= \frac{\left(-\frac{3}{2}\sqrt{\frac{3t}{t+1}}\right)\left(\frac{-3}{9t^2}\right)}{-\frac{1}{2}(t+1)^{-1/2}} = \frac{\left(+\frac{1}{2}\sqrt{\frac{3t}{t+1}}\right)\left(\frac{1}{t^2}\right)}{-\frac{1}{2\sqrt{t+1}}}$

$= -\frac{\sqrt{3t}}{t^2} = -\frac{\sqrt{3t}\sqrt{3t}}{(\sqrt{3t})(t^2)} = -\frac{3t}{(\sqrt{3t})(t^2)} = \boxed{-\frac{3}{t\sqrt{3t}}}$

OR $-\frac{\sqrt{3}}{3t\sqrt{t}}$

OR...

202 S'11.2#s 17, 18, 22, 26, 32, 35

#s 15-20 Implicit. Find slope @ given t-val.

18 $x\sin t + 2x = t, \quad t\sin t - 2t = y \quad t = \pi$

$x'\sin t + x\cos t + 2x' = 1$

$x' = \dfrac{1 - x\cos t}{\sin t + 2}$

$\sin t + t\cos t - 2 = y'$

@ $t = \pi$:

$x\sin\pi + 2x = \pi \qquad \pi\sin\pi - 2\pi = y$

$\boxed{x = \frac{\pi}{2}} \qquad \boxed{y = -2\pi}$

$\therefore \left.\dfrac{dy}{dx}\right|_{t=\pi} = \dfrac{\sin\pi + \pi\cos\pi - 2}{\dfrac{1 - \frac{\pi}{2}\cos\pi}{\sin\pi + 2}}$

$= \dfrac{-\pi - 2}{\dfrac{1 + \frac{\pi}{2}}{2}} = \left(\dfrac{2}{1 + \frac{\pi}{2}}\right)(-\pi - 2) = \left(\dfrac{2}{\frac{2+\pi}{2}}\right)(-\pi - 2)$

$= -\dfrac{4(\pi + 2)}{\pi + 2} = \boxed{-4 = \left.\dfrac{dy}{dx}\right|_{t=\pi}}$

202 §11.2 #s 17, etc

17 $x+2x^{3/2}=t^2+t$

$y\sqrt{t+1}+2t\sqrt{y}=4$, $t=0$

$t=0$: $x+2x^{3/2}=0$

$x(1+2x^{\frac{1}{2}})=0$

$\boxed{x=0}$

$1+2x^{\frac{1}{2}}=0$ requires

$2x^{\frac{1}{2}}=-1$

$x>0$

$y(\sqrt{1})+0=4$

$\boxed{y=4}$

$\boxed{(x,y)=(0,4)}$

$x'+2\left(\frac{3}{2}\right)x^{\frac{1}{2}}x'=2t+1$

$x'=\dfrac{2t+1}{3\sqrt{x}+1}=\dfrac{dx}{dt}$

$y'(t+1)^{\frac{1}{2}}+y\left(\frac{1}{2}(t+1)^{-\frac{1}{2}}\right)+2\sqrt{y}+2t\left(\frac{1}{2}y^{-\frac{1}{2}}\right)y'=0$

$y'(t+1)^{\frac{1}{2}}+\frac{1}{2}y(t+1)^{-\frac{1}{2}}+2y^{\frac{1}{2}}+ty^{-\frac{1}{2}}y'=0$

$y'=\dfrac{-\frac{1}{2}y(t+1)^{-\frac{1}{2}}-2y^{\frac{1}{2}}}{(t+1)^{1/2}+ty^{-\frac{1}{2}}}=\dfrac{dy}{dt}$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$

$\left.\dfrac{dy}{dx}\right|_{t=0}=\left.\left(\dfrac{-\frac{1}{2}y(t+1)^{-\frac{1}{2}}-2y^{\frac{1}{2}}}{(t+1)^{\frac{1}{2}}+ty^{-\frac{1}{2}}}\right)\left(\dfrac{3\sqrt{x}+1}{2t+1}\right)\right|_{t=0}=$

$\left(\dfrac{-\frac{1}{2}(4)(1)-2(2)}{1+0}\right)\left(\dfrac{3\cdot0+1}{2\cdot0+1}\right)=\dfrac{-6}{1}=\boxed{-6=\dfrac{dy}{dx}}$

202 § 11-2 #s 22, 26, 32, 35

(22) Area enclosed by the y-axis and

$x = t - t^2,\ y = 1 + e^{-t}$

$\Rightarrow e^{-t} = y - 1 \qquad \Rightarrow dy = -e^{-t}dt$

$\Rightarrow -t = \ln(y-1)$

$t = -\ln(y-1)$

$\Rightarrow x = (-\ln(y-1)) - (-\ln(y-1))^2$

$= -\ln(y-1) - \ln(y-1)^2$

$= -\ln(y-1)[1 + \ln(y-1)] = g(y)$

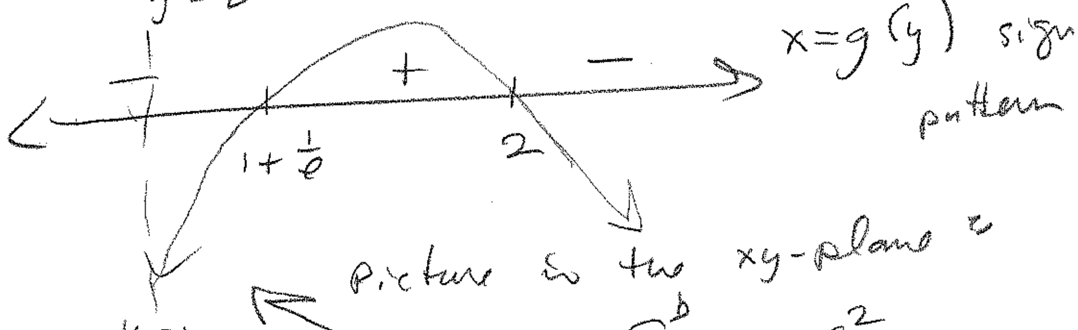
$-\ln(y-1) = 0 \qquad \ln(y-1) = -1$

$\ln(y-1) = 0 \qquad y - 1 = e^{-1} = \frac{1}{e}$

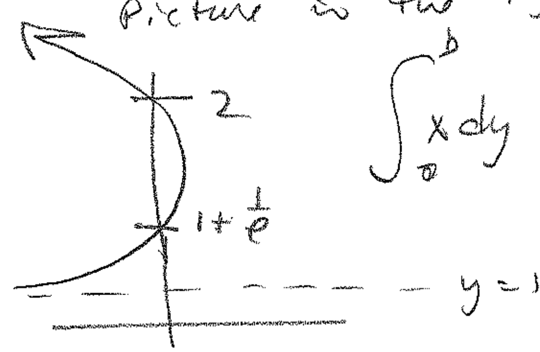
$y - 1 = e^0 = 1 \qquad y = 1 + \frac{1}{e}$

$y = 1 + 1 = 2$

$y = 2$



Picture in the xy-plane:



$\int_a^b x\,dy = \int_{1+\frac{1}{e}}^{2} (t - t^2)(-e^{-t}dt)$

202 §11.2 #s 22, 26, 32, 35

(22) Another go!

Area enclosed by y-axis &

$x = -t^2 + t$ and $y = 1 + e^{-t}$

$e^{-t} = y - 1$

$\ln(e^{-t}) = \ln(y-1)$

$-t = \ln(y-1)$

$t = -\ln(y-1)$

$x = -(-\ln(y-1))^2 + (-\ln(y-1))$

$= -(\ln(y-1))^2 - \ln(y-1)$

$= -\ln(y-1)(\ln(y-1)+1) \overset{\text{SET}}{=} 0$

$\ln(y-1) = 0$

$y - 1 = e^0 = 1$

$y = 2$ ✓

$\ln(y-1) + 1 = 0$

$\ln(y-1) = -1$

$y - 1 = \frac{1}{e}$

$y = 1 + \frac{1}{e}$

The BOOK solution

is $\int_0^1 x\,dy$ and NOT

$\int_{1+\frac{1}{e}}^{2} x\,dy$. I'm not grokking in fullness, here.

OH, WAIT! Integrating wrt t!

$y = 1 + \frac{1}{e} = 1 + \frac{1}{e^t} \Rightarrow \boxed{t = 1}$

$y = 2 = 1 + e^{-t} \Rightarrow e^{-t} = 1 \Rightarrow \boxed{t = 0}$

Now it works out fine

$\int_0^1 (t^2 - t)e^{-t}$

202 § 11.2 #s 22, 26, 32, 35

(22) cnt'd

$$\begin{array}{ccc} u & & dv \\ t^2 & + & e^{-t} \\ 2t & - & -e^{-t} \\ 2 & + & e^{-t} \\ 0 & & -e^{-t} \end{array} \qquad \begin{array}{ccc} u & & dv \\ -t & + & e^{-t} \\ -1 & - & -e^{-t} \\ 0 & & e^{-t} \end{array}$$

Teacher needs to spread out the columns.

$$\int_0^1 (t^2 - t)e^{-t}\,dt = \left[-t^2e^{-t} - 2te^{-t} - 2e^{-t} + te^{-t} + e^{-t}\right]_0^1$$

$$= \left[-t^2e^{-t} - te^{-t} - e^{-t}\right]_0^1 = \left(-1e^{-1} - 1e^{-1} - e^{-1}\right) - \left(-e^{-0}\right)$$

$$= -\frac{3}{e} + 1 = \boxed{1 - \frac{3}{e}}$$

(26) Find L: $x = t^3$, $y = \frac{3t^2}{2}$, $0 \le t \le \sqrt{3}$

$\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 3t$

$\left(\frac{dx}{dt}\right)^2 = 9t^4$ $\quad \left(\frac{dy}{dt}\right)^2 = 9t^2$

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\sqrt{3}} \sqrt{9t^4 + 9t^2}\,dt$$

$9t^4 + 9t^2$
$= 9t^2(t^2 + 1)$

$$= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\,dt = I$$

or let $u = t^2 + 1 \rightarrow$
$du = 2t\,dt$
~~$u(0) = 1$, $u(\sqrt{3}) = 4$~~

$$= \frac{3}{2}\int_0^{\sqrt{3}} (t^2+1)^{\frac{1}{2}}(2t\,dt) = \frac{3}{2}\left[\frac{2}{3}(t^2+1)^{3/2}\right]_0^{\sqrt{3}} = \boxed{7}$$

$4^{\frac{3}{2}} - 1 = 8 - 1 = 7$

202 § 11.2 #s 32, 35

Surface Area

32 $x = \frac{2}{3}t^{\frac{3}{2}}, y = 2t^{\frac{1}{2}}$ $0 \le t \le \sqrt{3}$, y-axis

$\int 2\pi y \, ds$

$\frac{dx}{dt} = t^{\frac{1}{2}} \Rightarrow \left(\frac{dx}{dt}\right)^2 = t$

$\frac{dy}{dt} = t^{-\frac{1}{2}} \rightarrow \left(\frac{dy}{dt}\right)^2 = \frac{1}{t}$

Surface area $= 2\pi \int_0^{\sqrt{3}} 2t^{\frac{1}{2}} \sqrt{t + \frac{1}{t}} \, dt$

$t = \tan\theta$
$dt = \sec^2\theta \, d\theta$
$t = 0 \Rightarrow \theta = 0$
$t = \sqrt{3} \rightarrow \theta = \frac{\pi}{3}$

$= 2\pi \int_0^{\sqrt{3}} 2\sqrt{t^2+1} \, dt$

2, $\sqrt{3}$, 1

$= 2\pi \int_0^{\frac{\pi}{3}} 2\sqrt{\tan^2\theta + 1} \, \sec^2\theta \, d\theta$

$= 2\pi \cdot 2 \int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta = 4\pi \left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta| \right]_0^{\frac{\pi}{3}}$

$= 2\pi \left[(2)(\sqrt{3}) + \ln|2 + \sqrt{3}| - (0 + 0) \right]$

$= \boxed{4\sqrt{3}\pi + 2\pi \ln|\sqrt{3} + 2|}$

Beautiful job on x-axis rotation

But asked for y-axis!

202 §11.2 #s 32, 35

(32) THIS time, revolve around the y-axis

$$2\pi \int_a^b x \, ds = 2\pi \int_0^{\sqrt{3}} \tfrac{2}{3} t^{\frac{3}{2}} \sqrt{t + \tfrac{1}{t}} \, dt$$

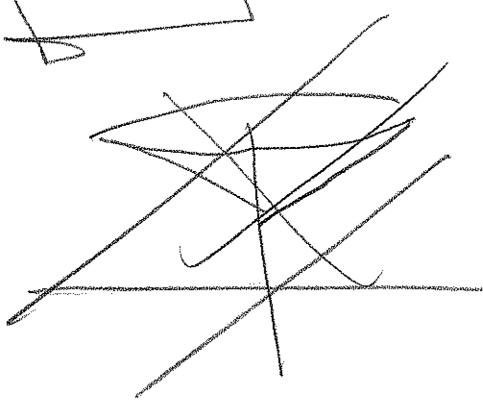
$$= 2\pi \int_0^{\sqrt{3}} \tfrac{2}{3} t \cdot t^{\frac{1}{2}} \sqrt{t + \tfrac{1}{t}} \, dt$$

$$= \frac{4\pi}{3} \int_0^{\sqrt{3}} t \sqrt{t^2+1} \, dt = \frac{4\pi}{3} \cdot \frac{1}{2} \int_0^{\sqrt{3}} (t^2+1)^{\frac{1}{2}} (2t \, dt)$$

$$= \frac{2\pi}{3} \cdot \frac{2}{3} \left[(t^2+1)^{\frac{3}{2}} \right]_0^{\sqrt{3}} = \frac{4\pi}{9} \left[4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right]$$

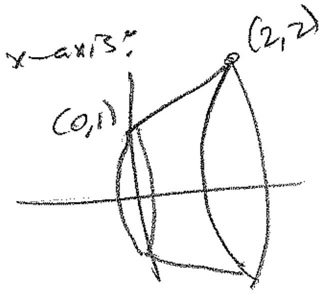
$$= \frac{4\pi}{9} [8 - 1] = \boxed{\frac{28\pi}{9}}$$

$y = \frac{1}{2} x + 1$



(35) A cone frustum

x-axis:



Use $x = 2t$, $y = t + 1$

$0 \le t \le 1$

How do they come up with these parametrizations?

You'll see a lot more of this in CALC III?

Want $(x, y) = (0, 1)$ when $t = 0$

$(x, y) = (2, 2)$ when $t = 1$

$x = \underset{\substack{\text{START} \\ t=0}}{(1-t)(0)} + \underset{\substack{\text{END} \\ t=1}}{t(2)} = 2t$

$y = \underset{\substack{\text{start} \\ t=0}}{(1-t)(1)} + \underset{\substack{\text{end} \\ t=1}}{t(2)}$

$= 1 - t + 2t = t + 1 = y$

See?

202 §11.2 #35 cont'd.

(35) $A = 2\pi \int_0^1 y\,ds = 2\pi \int_0^1 (t+1)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= 2\pi \int_0^1 (t+1)\sqrt{4+1}\,dt = 2\sqrt{5}\,\pi \int_0^1 (t+1)\,dt$

$= 2\sqrt{5}\,\pi \left[\frac{1}{2}t^2+t\right]_0^1 = 2\sqrt{5}\,\pi\left[\frac{1}{2}(1)^2+(1)-(0+0)\right]$

$= 2\sqrt{5}\,\pi\left[\frac{1}{2}+1\right] = \boxed{3\sqrt{5}\,\pi}$

Check: Area $= \boxed{\pi(r_1+r_2)(\text{slant height})}$

$= 2\pi\left(\frac{r_1+r_2}{2}\right)(\text{slant height})$ Derivation of surface area integral

$= 2\pi\left(\begin{matrix}\text{Avg}\\ \text{radius}\end{matrix}\right)(\text{arc length})$

$= 2\pi\ y\ ds$

$= \pi(1+2)(\sqrt{5}) = 3\sqrt{5}\,\pi$ ~~Not quite~~

$d(P,Q) = \sqrt{(2-0)^2+(2-1)^2}$ $\sqrt{4+1} = \sqrt{5}$