202 §10.5 #s 12, 14, 15, 17, 21, 27, 33, 45, 57, 61

#s 9–16 Root Test

(12) $\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1} = \sum a_n$

$$\sqrt[n]{|a_n|} = \left(\ln\left(e^2+\frac{1}{n}\right)^{n+1}\right)^{\frac{1}{n}}$$

$$= \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{\frac{n+1}{n}} \xrightarrow{n\to\infty} \left(\ln\left(e^2\right)\right)^1 = 2 > 1$$

$\Rightarrow \boxed{\text{Diverges}}$

(14) $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

$$\sqrt[n]{|a_n|} = \left(\left(\left|\sin\left(\frac{1}{\sqrt{n}}\right)\right|\right)^n\right)^{\frac{1}{n}} = \left|\sin\left(\frac{1}{\sqrt{n}}\right)\right| \xrightarrow{n\to\infty} 0 < 1$$

$\Rightarrow \boxed{\text{Converges}}$

(15) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2} = \sum a_n$

$$\sqrt[n]{|a_n|} = \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \left(1-\frac{1}{n}\right)^n = y \Rightarrow$$

$$\ln y = n\ln\left(1-\frac{1}{n}\right) \xrightarrow{n\to\infty} \infty\cdot 0$$

$$= \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow[L'H]{n\to\infty} \frac{\frac{\frac{1}{n^2}}{1-\frac{1}{n}}}{-\frac{1}{n^2}} = -\frac{1}{1-\frac{1}{n}} \xrightarrow{n\to\infty} -1$$

$\Rightarrow \lim_{n\to\infty} y = e^{-1} = \frac{1}{e} < 1 \Rightarrow \boxed{\text{Converges!}}$

202 §10.5 #s 3, 7, 12, 14, 15, 17, 21, 27, 33, 45, 57, 61

#s 1-8 RATIO TEST

3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ Instinct: $(n-1)!$ grows quicker than $(n+1)^2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+2)^2} \cdot \frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$= \frac{n \cdot n^2 (1+\frac{1}{n})^2}{n^2(1+\frac{2}{n})^2} = \frac{(1+\frac{1}{n})^2}{(1+\frac{2}{n})^2} n \xrightarrow{n\to\infty} \infty$$

$\boxed{\text{Diverges}}$

7. $\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}}$ Instinct: Converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2 (n+3)!}{(n+1)!\, 3^{2(n+1)}} \cdot \frac{n!\, 3^{2n}}{n^2 (n+2)!}$$

$$= \frac{(n+1)^2}{n^2} \cdot \frac{(n+3)!}{(n+2)!} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^{2n}}{3^{2n+2}}$$

$$= \frac{(n+1)^2}{n^2} \cdot (n+3) \cdot \frac{1}{(n+1)} \cdot \frac{1}{3^2} = \frac{(n+1)^2 (n+3)}{n^2 (n+1) 3^2} \xrightarrow{n\to\infty} \frac{1}{9} < 1$$

$\Rightarrow$ Converges

202 §10.5 #s 17, 21, 27, 33, 45, 57, 61

#s 17-44 Any Method

(17) $\sum \frac{n^{\sqrt{2}}}{2^n}$ Exponentials grow faster than power funcs

$$\sqrt[n]{|a_n|} = \left(\frac{n^{2^{\frac{1}{2}}}}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{\sqrt{2}}{n}}}{2^{\frac{n}{n}}} = \frac{n^{\frac{\sqrt{2}}{n}}}{2}$$

$$\lim_{n\to\infty} n^{\frac{\sqrt{2}}{n}} = \lim_{n\to\infty} y \Rightarrow$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\sqrt{2}}{n} \cdot \ln n = 0 \cdot \infty$$

$$= \lim_{n\to\infty} \sqrt{2}\left(\frac{\ln n}{\frac{1}{n}}\right) \overset{L'H}{=} \sqrt{2} \lim_{n\to\infty} \frac{\frac{1}{n}}{-\frac{1}{n^2}}$$

$$= -\sqrt{2} \lim_{n\to\infty} \frac{n^2}{n} = \infty \Rightarrow$$

$$\lim_{n\to\infty}(\ln y) = \infty \text{, so } \lim_{n\to\infty} y = \infty$$

$\boxed{\text{Diverges.}}$

$\left(\frac{\text{power}}{\text{exponen}} \longrightarrow 0\right)$

(21) $\sum \frac{n^{10}}{10^n}$

$$\sqrt[n]{|a_n|} = \left(\frac{n^{10}}{10^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{10}{n}}}{10^{\frac{n}{n}}} = \frac{n^{\frac{10}{n}}}{10} =$$

$$= \frac{1}{10} n^{\frac{10}{n}}$$

$y = n^{\frac{10}{n}}$

$\ln y = \frac{10}{n} \ln n \xrightarrow{n\to\infty} \frac{\frac{1}{n}}{-\frac{10}{n^2}} = -\frac{n}{10} \longrightarrow -\infty?$

Doesn't seem right

202 §10.5 #s 21, 27, 33, 45, 57, 61

(21) cntd

$\frac{1}{10} n^{\frac{10}{n}} = y$

$\ln(y) = \ln\left(\frac{1}{10} n^{\frac{10}{n}}\right) = \ln\left(\frac{1}{10}\right) + \frac{10}{n}\ln(n)$

$\xrightarrow{n\to\infty} 0 \cdot \infty$, so, re-write

$-\ln 10 + \frac{\ln(n)}{\frac{n}{10}} \xrightarrow[L'H]{n\to\infty} -\ln 10 + \frac{\frac{1}{n}}{\frac{1}{10}} = \frac{10}{n} - \ln 10 \xrightarrow{n\to\infty} -\ln 10$

$\Rightarrow \lim_{n\to\infty} \ln y = -\ln 10 \Rightarrow$

$\lim_{n\to\infty} y = e^{\ln\left(\frac{1}{10}\right)} = \frac{1}{10} < 1 \Rightarrow$ $\boxed{\text{converges}}$

∞

(27) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3} = \sum a_n$

$\ln n < n \ \forall n \geq 1 \Rightarrow \frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$

$\Rightarrow \sum b_n = \sum \frac{1}{n^2}$ is comparison

p-test $\Rightarrow \sum b_n$ converges $\Rightarrow \sum a_n$ $\boxed{\text{converges}}$

(33) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = \frac{(n+2)(n+3)}{(n+1)(n+2)} \cdot \frac{1}{n}$

$\xrightarrow{n\to\infty} 0 \Rightarrow$ $\boxed{\text{converges}}$

202 §10.5 ~~4~~5 45, 57, 61

45 ~~R~~ Recursives

$a_1 = 2, \quad a_{n+1} = \frac{1+\sin n}{n} a_n$

$a_2 = \frac{1+\sin 2}{2} \cdot 2 = 1+\sin 2$

$a_3 = \frac{1+\sin 3}{3} \cdot (1+\sin 2)$

$a_4 = \frac{1+\sin 4}{4} \cdot \frac{1+\sin 3}{3} \cdot (1+\sin 2)$

$\vdots$

$a_n = \frac{1+\sin n}{n} \cdot \frac{1+\sin(n-1)}{n-1} \cdots \frac{1+\sin 3}{3} \cdot (1+\sin 2)$

$= \left(\prod_{k=3}^{n} \frac{1+\sin k}{k}\right)\left(\frac{1+\sin 2}{2} \cdot 2\right)$

$= \left(2 \prod_{k=2}^{n} \frac{1+\sin k}{k}\right) = \frac{2}{n!} \prod_{k=2}^{n} (1+\sin k)$

$\leq \frac{2}{n!} \prod_{k=2}^{n} 2 = \frac{2}{n!} 2^{n-1} = \frac{2^n}{n!}$

Ratio on $\sum b_n = \sum \frac{2^n}{n!}$:

$\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \xrightarrow{n \to \infty} 0 \longrightarrow \sum b_n$ converges.

Since $a_n \leq b_n$, $a_n \geq 0$, $\sum a_n$ converges

201 §10.5 #s 57, 61

Converge? Back it up.

57 $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}} = \sum a_n$

Root: $\sqrt[n]{\frac{(n!)^n}{n^{2n}}} = \frac{n!}{n^2} \xrightarrow{n\to\infty} \infty$

$$\frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n^2} = \frac{n^2(1-\frac{1}{n})(1-\frac{2}{n})(n-3)\cdots(2)(1)}{n^2}$$

$$= (1-\tfrac{1}{n})(1-\tfrac{2}{n})(n-3)\cdots(3)(2)(1) \xrightarrow{n\to\infty} \infty \checkmark$$

$\Rightarrow \sum a_n$ Diverges, by Ratio Test.

61 $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{4^n \cdot 2^n n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1\cdot 3\cdots(2n-1)(2(n+1)-1)}{4^{n+1}2^{n+1}(n+1)!} \cdot \frac{4^n \cdot 2^n \cdot n!}{1\cdot 3\cdots(2n-1)}$$

$$= \frac{2n+2-1}{4\cdot 2\cdot(n+1)} = \frac{2n+1}{8n+8} \xrightarrow{n\to\infty} \frac{2}{8} = \frac{1}{4} < 1$$

$\Rightarrow$ $\boxed{\text{Converges}}$

202 §10.4 #s 1, 4, 7, 10, 18, 21, 24, 29, 43

#s 1-8 Comparison Test

① $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ $\qquad \frac{1}{n^2+30} > 0$ ✓

$a_n = \frac{1}{n^2+30} < \frac{1}{n^2} = b_n$ & $\sum b_n$ converges

$\Rightarrow \boxed{\sum a_n \text{ converges}}$

④ $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

$a_n = \frac{n+2}{n^2-n} \leq \frac{n+n}{n^2-\frac{1}{2}n^2} = \frac{2n}{\frac{1}{2}n^2} = \frac{\cancel{4}}{n}$ diverges

so, we want $a_n \geq$ something that diverges

I should've seen divergence right away

(p-test, $n=1$)

$a_n = \frac{n+2}{n^2-n} \geq \frac{n}{n^2} = \frac{1}{n} = b_n$

$\sum b_n$ diverges $\Rightarrow \boxed{\sum a_n \text{ diverges}}$

⑦ $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^2+4}}$ looks like $a_n \approx \sqrt{\frac{n}{n^2}} = \sqrt{\frac{1}{n}}$

$\sum \sqrt{\frac{1}{n}}$ diverges. so find something smaller

than a_n:

$a_n = \sqrt{\frac{n+4}{n^2+4}} \geq \sqrt{\frac{n}{n^2+\frac{1}{2}n^2}} = \sqrt{\frac{n}{\frac{1}{2}n^2}} = \frac{\sqrt{2}}{\sqrt{n}} = b_n$

$\sum b_n$ diverges $\Rightarrow \boxed{\sum a_n \text{ diverges}}$