§10.3

If $a_n \ge 0$ for all n, then

$$\{S_n\} = \{\sum_{k=1}^{n} a_k\}$$

$= \{a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\dots+a_n, \dots\}$ is increasing (nondecreasing). So, if the S_n's are bounded above, then $\sum_{k=1}^{\infty} a_k$, by Monotone Convergence Theorem.

Pg 553 $\sum \frac{1}{n}$ is the HARMONIC SERIES

Read the argument for why it diverges.

$$\underbrace{1 + \frac{1}{2} + \underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}}_{S_{2^2} > \frac{2}{2}} + \underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9}+\dots+\frac{1}{16}}_{>\frac{1}{2}} + \dots$$

$$S_{2^3} = S_8 > \frac{3}{2}$$

$$S_{2^4} = S_{16} > \frac{4}{2}$$

$$S_{2^k} > \frac{k}{2}$$

$$\sum \frac{1}{n^{1.0000001}}$$

So, S_n increases without Bound.

Want $S_n > 500 = \frac{1000}{2}$

$$S_{2^{1000}} > \frac{1000}{2} = 500$$

The Integral Test

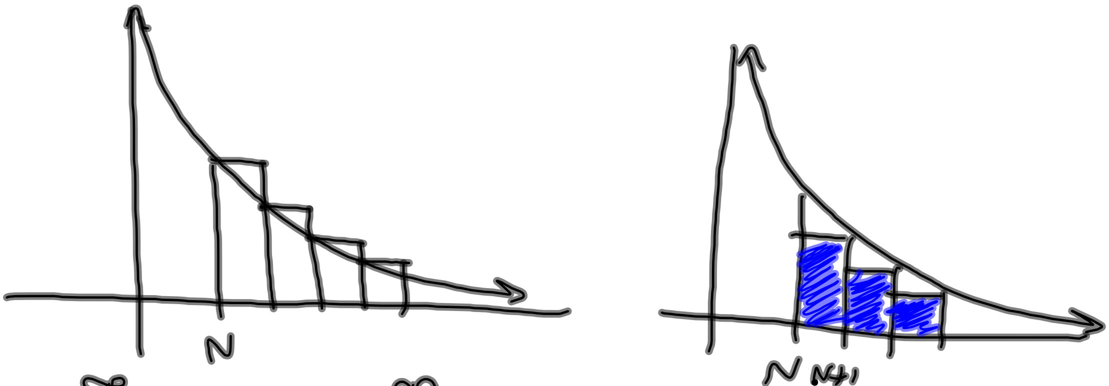
$\sum f(k)$ converges iff $\int_N^\infty f(x)\,dx$ converges

$\sum_{k=1}^\infty \frac{1}{k^2}$ converges, because $\int_1^\infty \frac{dx}{x^2}$ does

Necessary Hypotheses:

f(x) is continuous

f(x) is (eventually) decreasing.



$$\sum_{k=N}^\infty f(k) \geq \int_N^\infty f(x)\,dx \geq \sum_{k=N+1}^\infty f(k)$$

$$= f(N)\cdot 1 + f(N+1)\cdot 1 + \dots$$

$$\sum \frac{1}{n^2+1} \leq \sum \frac{1}{n^2}$$

$$S = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N} a_k + \underbrace{\sum_{k=N+1}^{\infty} a_k}_{N\text{-tail}} \longrightarrow$$

$$S - S_N = \sum_{k=N+1}^{\infty} a_k = R_N$$

By Integral Test Picture, we know:

$$\int_{N+1}^{\infty} f(x)\,dx \le \boxed{\sum_{k=N+1}^{\infty} a_k \le \int_{N}^{\infty} f(x)\,dx}$$

for f decreasing, continuous, for $x > N$

I want to estimate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ within $\pm .001$

How many terms will it take?

$$\text{Error} = R_N = \sum_{k=N+1}^{\infty} \frac{1}{k^2} \leq \int_N^{\infty} \frac{1}{x^2}\,dx \quad \overset{\text{MAKE}}{\leq} .001$$

So...

$$\lim_{t\to\infty} -x^{-1}\Big]_N^t = \lim_{t\to\infty}\left[-\frac{1}{t} - \left(-\frac{1}{N}\right)\right] = \frac{1}{N} \overset{\text{want}}{<} .001$$

$$\frac{1}{N} < .001 = \frac{1}{1000}$$

$1000 < N$ terms

How about within .01

... $N > 100$

$$evalf\left(\sum_{k=1}^{101} \frac{1}{k^2}\right)$$

$$1.635081930$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\frac{1}{6}\pi^2$$

$$evalf(\%)$$

$$1.644934068$$

$$evalf\left(\frac{1}{6}\pi^2 - 1.635081930\right)$$

$$0.009852138$$

§10.3 #5 6, 12, 18, 22, 29, 34, 49, 50, 52

Building-Up to representation of smooth functions with power series. (TAYLOR'S SERIES)

$$e^x = \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots} + \frac{x^n}{n!}$$

$$\frac{d^n}{dx^n}\left[x^n\right] = n \cdot (n-1) \cdot (n-2) \cdots$$

Turns e^x into a polynomial!

8 terms is like 11-digit accuracy.