$-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}$ Looks like a recursion...

$(-1)^n$

$a_1 = -\frac{3}{2}$

$a_2 = f(a_1)$?

$-\frac{1}{6} = f(-\frac{3}{2})$

$\frac{1}{12} = f(-\frac{1}{6})$

$-2 + f(n)$

$\frac{5}{30} - \frac{3}{20} = \frac{10-9}{60} = \frac{1}{60} > 0$

$\frac{3}{20} - \frac{1}{12} =$

$\frac{3}{3} \cdot \frac{3}{2 \cdot 2 \cdot 5} - \frac{1}{2 \cdot 2 \cdot 3} \cdot \frac{5}{5} = \frac{1}{60} > 0$

Treat numerator and denominator as separate sequences:

$-3, -1, 1, 3, 5, \ldots$

$n=1$: $-3 = 1-4$ $n-4$

$2(1)-5$ $2n-5$

$n=2$: $-1 = 2-4$? Nope

$-1 = 2(2)-5$? Yep

$n=3$: $2(3)-5=1$ ✓

$n=4$: $2(4)-5=3$ ✓

$n=5$: $2(5)-5=5$ ✓

$2\cdot1$ $2\cdot3$ $2\cdot2\cdot3=3\cdot4$ $2\cdot2\cdot5=4\cdot5$ $2\cdot3\cdot5=5\cdot6$

$2, 6, 12, 20, 30$

In real life, this might be my $10^{th}$ guess.

$n=1$: $2n$

$\boxed{n(n+1)}$

$1(1+1)=2$ ✓

$n=2$: $2(2)=4 \neq 6$

$2(3)=6$ ✓

$3(4)=12$ ✓

You guys are good ⁝

$\frac{2n-5}{n(n+1)}$ seems to do the trick!

10.2

Does $\sum \left(\sqrt{n+4} - \sqrt{n+3}\right)$ converge?

I guess NO. Test for DIVERGENCE

$\lim_{n\to\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right) = 0$? If not, then diverges.

$$\left(\sqrt{n+4} - \sqrt{n+3}\right)\left(\frac{\sqrt{n+4} + \sqrt{n+3}}{\sqrt{n+4} + \sqrt{n+3}}\right) \qquad \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{n+4-(n+3)}{\sqrt{n+4} + \sqrt{n+3}} = \frac{1}{\sqrt{n+4} + \sqrt{n+3}} \xrightarrow{n\to\infty} 0$$

So no easy "No," here. Investigate further:

$$\sqrt{1+4} - \sqrt{1+3} + \sqrt{2+4} - \sqrt{2+3} + \cdots$$

$$S_7 = \sqrt{5} - \sqrt{4} + \sqrt{6} - \sqrt{5} + \sqrt{7} - \sqrt{6} + \sqrt{8} - \sqrt{7} +$$

$$+ \sqrt{9} - \sqrt{8} + \sqrt{10} - \sqrt{9} + \sqrt{11} - \sqrt{10}$$

$$= -2 + \sqrt{7+4}$$

$$\boxed{S_n = -2 + \sqrt{n+4}}$$

Nice, closed-form expression for the $n^{\text{th}}$ partial sum.

We test $\{S_n\}$ for convergence

But $S_n \xrightarrow{n\to\infty} \infty$

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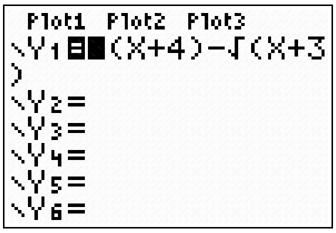


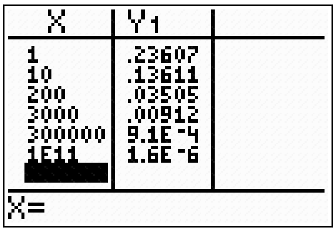


Numerical Exploration of $\sqrt{n+4} - \sqrt{n+3}$

Let these guide your intuition.

$\boxed{\sum \frac{1}{n}}$ also "passes" test for divergence

$\frac{1}{n} \xrightarrow{n \to \infty} 0$, but FAILS p-test

$\frac{1}{n^1}$ p must be > 1 to converge:

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx}$$

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Finding Closed-Form Expressions for $S_n$ is not generally an easy thing.

Finding the Actual Value of $\sum a_n$ is typically very elusive / impossible.

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other facts:

Adding / Deleting a finite number of terms has NO bearing on convergence questions.

Re-indexing for convenience changes nothing

$$\sum_{k=6}^{\infty} 7\cdot\left(\tfrac{2}{3}\right)^k = 7\sum_{k=6}^{\infty}\left(\tfrac{2}{3}\right)^k = 7\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)^k - 7\sum_{k=1}^{5}\left(\tfrac{2}{3}\right)^k$$

$$= 7\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)^{k-1} - \boxed{7\sum_{k=1}^{5}\left(\tfrac{2}{3}\right)^k}$$

$a = \frac{2}{3}$

$r = \frac{2}{3}$

a finite sum we can use brute force to compute.

Jon says:

$$\sum_{k=6}^{\infty} 7\left(\tfrac{2}{3}\right)^k = 7\sum_{k=6}^{\infty}\left(\tfrac{2}{3}\right)^k = 7\left[\left(\tfrac{2}{3}\right)^6 + \left(\tfrac{2}{3}\right)^7 + \cdots\right]$$

$$= 7\left[\overset{1}{\left(\tfrac{2}{3}\right)^6} + \overset{2}{\left(\tfrac{2}{3}\right)^6\left(\tfrac{2}{3}\right)} + \overset{3}{\left(\tfrac{2}{3}\right)^6\left(\tfrac{2}{3}\right)^2} + \cdots\right]$$

$$= 7\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)^6\left(\tfrac{2}{3}\right)^{k-1}$$

$a = \left(\frac{2}{3}\right)^6$

$r = \frac{2}{3}$

Goal: write $\sum_{k=6}^{\infty}$ as $\sum_{k=1}^{\infty}$

$$S = 7\cdot\frac{\left(\frac{2}{3}\right)^6}{1-\frac{2}{3}} = 7\cdot\frac{a}{1-r}$$

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{(n+1)(n+2)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} \qquad \left( = \sum_{k=2}^{\infty} \frac{1}{k(k+1)} \right)$$

Partial fracs: $\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$\Longrightarrow A = 1, B = -1$, giving

$$\sum_{k=1}^{\infty} \left[ \frac{1}{k+1} - \frac{1}{k+2} \right] = \underline{\frac{1}{2} - \frac{1}{3}} + \underline{\frac{1}{3} - \frac{1}{4}} + \underline{\frac{1}{4} - \frac{1}{5}} + \cdots$$

$$S_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{n+2} \xrightarrow{n \to \infty} \frac{1}{2}$$

$$\{S_n\} = \left\{ \frac{1}{2} - \frac{1}{3}, \frac{1}{2} - \frac{1}{4}, \frac{1}{2} - \frac{1}{5}, \frac{1}{2} - \frac{1}{6}, \ldots \right\}$$

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