§10.1

$a_n = f(n) = \sqrt{n}$ a function whose domain is $\mathbb{N} = \{1,2,3,\dots\}$

$= \{1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\}$

$= \{\sqrt{n}\} = \{\sqrt{n}\}_{n=1}^{\infty}$

Convergence / Divergence.

$b_k = \frac{k-1}{k}$, $k = 1,2,\dots$

$= \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

The sequel: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$

Fails p-test

$\int_1^{\infty} \frac{dx}{x} = \infty$

$a_n = \sum_{k=1}^{n} \frac{1}{k}$

$= \{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \dots\}$

$a_n = (-1)^{n+1}\frac{1}{n}$

$= \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\}$

In sequel, we'll see that

$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ Does converge!

$\boxed{D}$ Convergence:

$\lim_{n\to\infty} a_n = L$ means $\underbrace{\text{for any } \varepsilon > 0}_{\text{any challenge}}$ there is an $N > 0$ such that

$|a_n - L| < \varepsilon$ for all $n > N$.

$\boxed{E1}$ $\lim_{n\to\infty} \frac{1}{n} = 0$

Scratch:

want $|\frac{1}{n} - 0| < \varepsilon$

$\Leftrightarrow \frac{1}{n} < \varepsilon$

$\Leftrightarrow \frac{1}{\varepsilon} < n$, i.e.,

$\Leftrightarrow n > \frac{1}{\varepsilon} \equiv N$

Proof

Let $\varepsilon > 0$. Define $N = \frac{1}{\varepsilon}$.

Let $n > N$.

Then $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ ∎

$\boxed{E}$ $\lim_{n\to\infty} \frac{n-1}{n} = 1$

Scratch: want

$|\frac{n-1}{n} - 1| < \varepsilon$

$|\frac{n-1}{n} - \frac{n}{n}| < \varepsilon$

$|\frac{n-1-n}{n}| < \varepsilon$

$|-\frac{1}{n}| < \varepsilon$

$\frac{1}{n} < \varepsilon$

$N \equiv \frac{1}{\varepsilon} < n$

Proof:

Let $\varepsilon > 0$ be given

Define $N = \frac{1}{\varepsilon}$

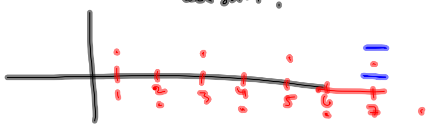
Then if $n > N$, we have

$|\frac{n-1}{n} - 1| = |-\frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ ∎

Read Example 2 §10.1

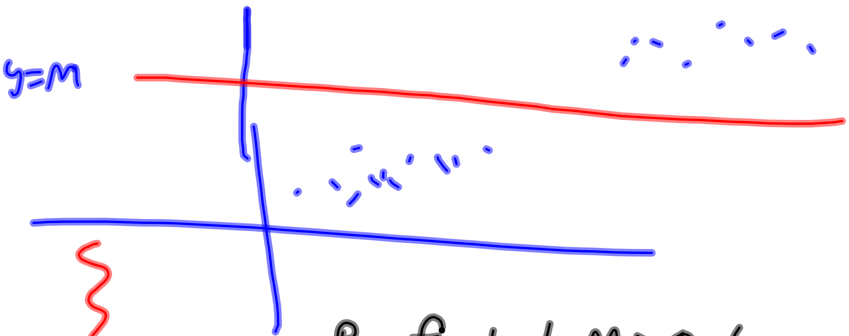
$a_n = (-1)^{n+1}$ Diverges.

$\varepsilon = \frac{1}{2}$ is enough to prove it doesn't.



$\lim_{n\to\infty} a_n = \infty$ if, for any $M > 0$, there is an N such that for all $n > N$, $a_n > M$.

No matter what the ceiling, Eventually all terms are above that ceiling



Show that $\lim_{n\to\infty} \sqrt{n} = \infty$

want $\sqrt{n} > M$

$n > M^2 \equiv N$

Proof Let $M > 0$ be given. Define $N = M^2$. Then if $n > N$, we have

$a_n = \sqrt{n} > \sqrt{N} = \sqrt{M^2} = M$ ∎

Sandwich Theorem (Squeeze Theorem)

If $a_n \le b_n \le c_n$ for all n and we know that

$a_n \to L$ & $c_n \to L$,

Then $b_n \to L$. Baloney!

$\boxed{E}$ $a_n = \frac{\sin n}{n}$ $|a_n| = \left|\frac{\sin n}{n}\right| \le \left|\frac{1}{n}\right| \xrightarrow{n\to\infty} 0$

using sammich theorem:

$-1 \le \sin n \le 1$

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$-\frac{1}{n} \to 0$, $\frac{1}{n} \to 0$

$\therefore \frac{\sin n}{n} \to 0$

Nonexample:

$a_n = 1$, $b_n = -1$

$a_n + b_n = 0$

$\lim_{n\to\infty}(a_n+b_n) = 0$, but neither a_n nor b_n converge.

$\boxed{T1}$ $a_n \to A$ & $b_n \to B$

1. $\lim(a_n+b_n) = \lim a_n + \lim b_n$
2. $\lim(a_n-b_n) = \lim a_n - \lim b_n$
3. k konstant $\Rightarrow$ $\lim(kb_n) = kB$
4. $\lim(a_nb_n) = AB = \lim a_n \lim b_n$
5. $\lim\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, provided $B \ne 0$.

Recursive Sequences

$a_1 = -1$, $a_{n+1} = \frac{a_n + 6}{a_n + 2}$ Find the limit!

$a_1 = -1$

$a_2 = 5$

$a_3 = \frac{11}{7}$

$a_4 = \frac{53}{25}$

$a_5 = \frac{203}{103} \approx 1.970873786$?

$a_6 = \frac{821}{409}$

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              53/25
Y1(Ans
        1.970873786
Y1(Ans
        2.007334963
Ans▸Frac
            821/409
█
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Guess $L = 2$

Suppose $|a_n - L| < \varepsilon$

$$\left| a_{n+1} - L \right| = \left| \frac{a_n + 6}{a_n + 2} - 2 \right| < \varepsilon$$

$$\left| \frac{a_n + 6 - 2a_n - 4}{a_n + 2} \right| < \varepsilon$$

$$\left| \frac{-a_n + 2}{a_n + 2} \right| < \varepsilon$$

etc. to see if we can get something here

S'10,1 #s 6, 10, 16, 18, 30, 36, 46, 92*

92 is DONE