$a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$

$a_2 = 3 - \frac{1}{1} = 2$

$a_3 = 3 - \frac{1}{2} = \frac{5}{2}$

$a_4 = 3 - \frac{2}{5} = \frac{13}{5}$

$a_4 = 3 - \frac{5}{13} = \frac{34}{13}$

WTS $a_{n+1} > a_n \ \forall n$ and Bdd above by 3.

They want a general argument, but spreadsheet is pretty compelling to a scientist.

$a_2 < 3$

$\S\ a_n < 3$

Then $a_{n+1} = 3 - \frac{1}{a_n} = \frac{3a_n - 1}{a_n}$

Show it's increasing.

$a_2 > a_1$

$\S\ a_{n+1} > a_n$

Then consider $a_{n+2} - a_{n+1}$

$= 3 - \frac{1}{a_{n+1}} - \left(3 - \frac{1}{a_n}\right)$

$= -\frac{1}{a_{n+1}} + \frac{1}{a_n} > 0$

Section 11.2 Series

Recall Sequence. $a_1, a_2, a_3, a_4, \ldots$

Now add all the terms $a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k$

in a sequence together!

π = 3.14159 26535 89793 23846 26433 83279 50288 . . .

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

Johann Friedrich Gauss

$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots + 100$

$$\left.\begin{matrix} 1+100 \\ 2+99 \\ 3+98 \\ 4+97 \\ \vdots \\ 49+52 \\ 50+51 \end{matrix}\right\} \quad 50(101) \quad 5050$$

$1 + 2 + \cdots + 100$

$= 50(101)$

$= \frac{100(101)}{2}$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Sequence of partial sums:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

$s_n = \sum_{k=1}^{n} a_k$

$\{s_n\}$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

If s_n converges,

Then $\sum_{k=1}^{\infty} a_k$ converges

$\{s_n\} = \left\{ \sum_{k=1}^{n} a_k \right\}$

Geometric Series (We see them in college algebra)

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Annuities, Immigration...

$$s_n = \frac{a(1-r^n)}{1-r}$$

we show this w/ telescope!

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

$$\sum_{n=1} \frac{1}{\sqrt{2}^{\,n-1}}$$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

Be mindful of where the series starts, before plugging into the formula!

with partial sum:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$-\;rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$(1-r)S_n = a - ar^n$$

$$\sum_{k=1}^{n} ar^{k-1} = S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum e\left(\frac{e^{n-1}}{3^{n-1}}\right) = e\sum_{n=1}^{\infty}\left(\frac{e}{3}\right)^{n-1}$$

NOTE: If $-1 < r < 1$ (i.e., $|r| < 1$)

then $r^n \xrightarrow{n \to \infty} 0$

So, if $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty}\left(\frac{a(1-r^n)}{1-r}\right) = \boxed{\frac{a}{1-r}}$$

0

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

$$S = \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

p-series $p=1$ series.

Recall the *p*-test for improper integrals? I'll never ask *you* to make this argument, but you should try to follow it. The Harmonic Series is the *p* = 1 case for infinite series, just the same as 1/*x* is the *p* = 1 case for improper integrals.

I'll make the argument, here, if time permits, but see Section 11.3 for more on this. Basically, *everything* we learned about improper integrals will apply to infinite series. Integral Test and related ideas are a great way to determine if a series converges, find upper and lower bounds for its sum, but, sadly, no quick, slick ways to determine just how big it is, *exactly*.

I think that's why I stayed a mathematician. It's enough to know it doesn't blow up, and I can move on, while the engineers figure out the 20th decimal place.

$$S = \sum_{K=1}^{\infty} \frac{1}{K} = \text{Harmonic Series Diverges.}$$

It diverges b/c it's bigger than something that we know diverges

(Trick to showing convergence is showing it's smaller than something that converges.)

$$S_1 = \frac{1}{1} = 1$$

$$S_2 = \frac{1}{1} + \frac{1}{2} = S_{2^1} = 1 + \frac{1}{2} = a_1$$

$$S_4 = S_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} = a_2$$

$$S_8 = S_{2^3} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) = 1 + \frac{3}{2} = a_3$$

$$S_{2^n} > a_n = 1 + \frac{n}{2} \xrightarrow{n \to \infty} \infty \text{ (i.e. Diverges!)}$$

$$S_{2^n} \xrightarrow{n \to \infty} \text{Diverges, too!}$$

p-test

$$\int_1^{\infty} \frac{dx}{x} = \lim_{n \to \infty} \int_1^n \frac{dx}{x}$$

$$\lim_{n \to \infty} \left[\ln(x)\Big|_1^n\right] =$$

$$\lim_{n \to \infty} \left[\ln(n)\right] = \cancel{\exists}$$

$$\int_1^{\infty} \frac{dx}{x^1} \longleftrightarrow \sum_{K=1}^{\infty} \frac{1}{K}$$

$$\int_1^{\infty} \frac{dx}{x^p} \quad \text{converges if } p > 1$$
$$\text{diverges if } p \le 1$$

$\boxed{6}$ **Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

$A \Rightarrow B$

The contrapositive of 6:

$\text{NOT } B \Rightarrow \text{NOT } A$

$\boxed{7}$ **The Test for Divergence** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Is Contrapositive

15. Let $a_n = \dfrac{2n}{3n+1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

$$a_n = \frac{2n}{3n+1} \xrightarrow{n\to\infty} \frac{2}{3} \neq 0$$

$\sum a_n$ Diverges.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

NOTE $\displaystyle\sum_{k=1}^{\infty}(1-1) = 0$

IS NOT $\sum 1 - \sum 1$

A word of caution: This only works the one direction. Sometimes the left hand sides will converge, but that doesn't the separate sums converge separately. Be very careful about confusing the converse with the original statement.

3–4 Calculate the sum of the series $\Sigma_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$ **4.** $s_n = \dfrac{n^2 - 1}{4n^2 + 1}$

5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

See video & Spreadsheet!

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 7). If it is convergent, find its sum.

43. $\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$ Partial Fractions

44. $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ $= \sum \left(\ln(n) - \ln(n+1) \right)$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots$

28. $\frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots$

29. $\sum_{n=1}^{\infty}\frac{n-1}{3n-1}$

30. $\sum_{k=1}^{\infty}\frac{k(k+2)}{(k+3)^2}$

3 $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}$

33. $\sum_{n=1}^{\infty}\sqrt[n]{2}$

34. $\sum_{n=1}^{\infty}[(0.8)^{n-1}-(0.3)^n]$