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11.1 Sequences

A **sequence** can be thought of as a list of numbers written $\boxed{\text{in a definite order:}}$

$\{1,2\} = \{2,1\}$ in general

$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $a_n = \frac{n}{n+1}$

a_n is nth element

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

(b) $\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots\}$

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

$3 \in \{1,2,3\}$

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

(c) $\{\sqrt{n-3}\}_{n=3}^{\infty}$ $a_n = \sqrt{n-3},\ n \geq 3$

$\{0, 1, \sqrt{2}, \sqrt{3}, 2, \ldots\}$

$\{\sqrt{n-3}\}_{\underline{n \in \mathbb{N}}} = \{1,2,3,\ldots\}$

$n = 1$ to ∞

(d) $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$ $a_n = \cos\frac{n\pi}{6},\ n \geq 0$

A sequence can be thought of as a function

$f : \mathbb{N} \longrightarrow \mathbb{R}$

$f(3) = 0$

$f(4) = 1$, etc.

$f(3) = a_3$

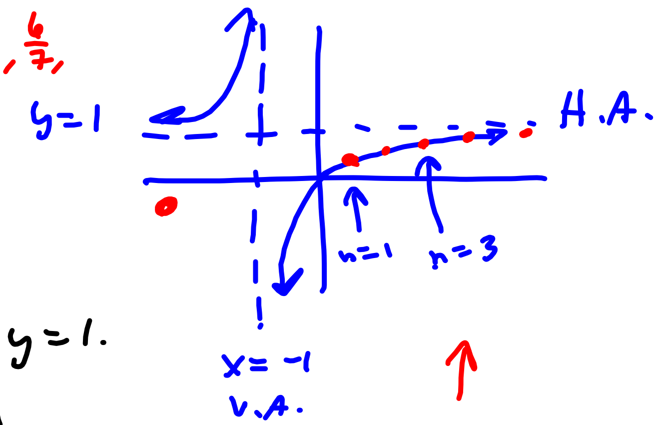
$a_n = \frac{n}{n+1} = f(n)$ $\{a_n\} = \{\frac{n}{n+1}\}$

converges

$f(n) = \frac{n}{n+1}$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$

$f(x) = \frac{x}{x+1} \xrightarrow{x \to \pm\infty} 1$



$\frac{n}{n+1} \xrightarrow{n \to \infty} 1$

converges to $y=1$.

$\lim_{n \to \infty} \frac{n}{n+1} = 1$

$\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots\}$ ellipsis

$n=1$: $\frac{2}{3} = \frac{n+1}{n+2} = \frac{n+1}{3^n}$

$n=1$: $\frac{1+1}{3^1} = \frac{2}{3}$

$n=2$: $\frac{2+1}{3^2} = \frac{3}{9}$

$n=2$: $\frac{n+1}{n+2} = \frac{3}{5}$ Nope.

$(-1)^n$: $-1, 1, -1, 1, -1, 1, -1, \dots$
has the right signs.

$\left\{(-1)^n \frac{n+1}{3^n}\right\} = \{a_n\}$

$\left\{\frac{2}{3}, -\frac{3}{9}, \frac{4}{27}, -\frac{5}{81}, \dots\right\} = \left\{(-1)^{n+1} \frac{n+1}{3^n}\right\}$
$(-1)^{n+1}$ can handle that.

A sequence such as the one in Example 1(a), $a_n = n/(n+1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2.

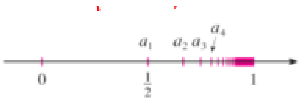


Figure 1

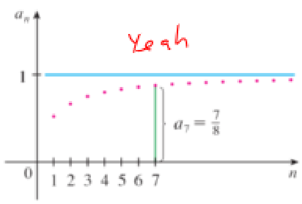
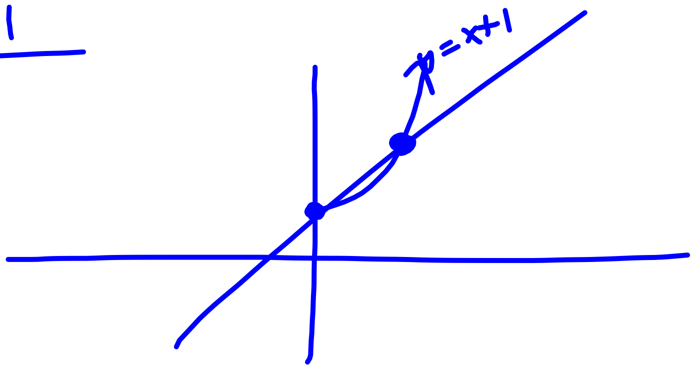


Figure 2

$f(x) = \frac{x}{x+1}$ informs this

Think $f(x) = \frac{x}{x+1}$ to get the picture.

$\frac{n+1}{3^n}$



1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty \qquad \text{OR} \qquad a_n \xrightarrow{n\to\infty} L$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

EVENTUALLY

$|a_n - L| <$ Small

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

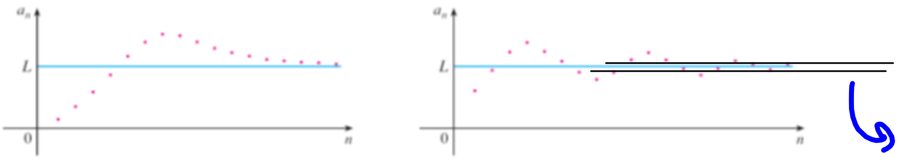
$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \varepsilon$$

scratch want $\left|1 - \frac{n}{n+1}\right| < \varepsilon$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$



↳ we in the ε-tube!

Figure 3

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$

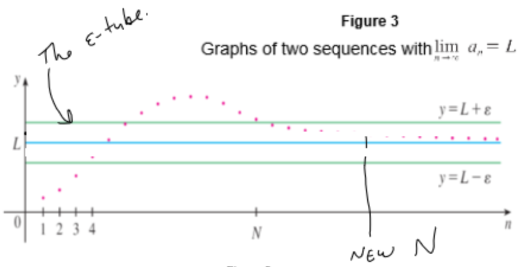


Figure 5

"Eventually" means there is an $N > 0$ such that

$|a_n - L| < \varepsilon$

(i.e. $L-\varepsilon < a_n < L+\varepsilon$)

yes in the tube, baby!

ε
epsilon

$\lim_{n\to\infty} a_n = L$ means

Given $\varepsilon > 0$, $\exists N > 0 \ni$

$\boxed{|a_n - L| < \varepsilon}\ \forall n > N$

To say that $a_n \xrightarrow{n\to\infty} L$ is to say that

Give me an $\varepsilon > 0$

I can find an N such that for every $n > N$, we will have

$|a_n - L| < \varepsilon$.

$-\varepsilon < a_n - L < \varepsilon$

$L - \varepsilon < a_n < L + \varepsilon$

$\forall$ - for all, for each, for every

$\ni$ - so that, such that.

$\exists$ - there is

Formal Definition:

$\lim_{n\to\infty} a_n = L$ means.

given any $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$

$\forall n > N$, we have

$|a_n - L| < \varepsilon$.

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

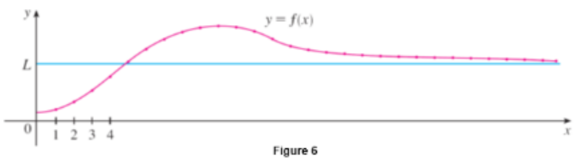
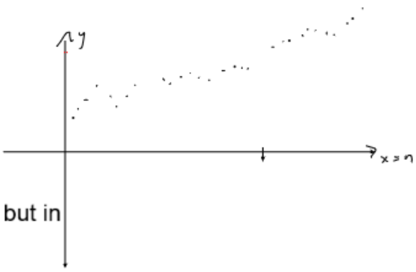


Figure 6

In particular, since we know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$, we have

$$\boxed{4} \qquad \lim_{x\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. Consider the definition



5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

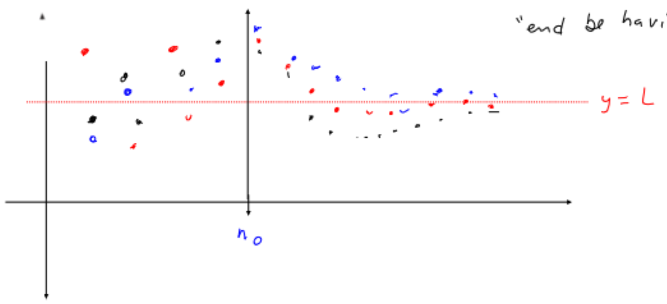
Be careful. Sometimes $\{a_n\}$ &/or $\{b_n\}$ don't converge, separately. BOLO for this.

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

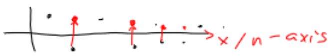
Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

$\lim_{n\to\infty} a_n$ is all about "eventually," "end behavior."



6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.



7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

F cnt^s means $\lim_{x\to a} f(x) = f(a)$

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$r = .2$

$.2, .2^2, .2^3, .2^4, \ldots \ldots, .2^n, \ldots$

is monotone decreasing

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

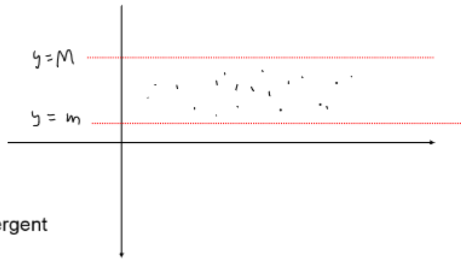
11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.



not every bounded sequence is convergent
$a_n = (-1)^n$

not every monotonic sequence is convergent ($a_n = n \to \infty$).

But if a sequence is both bounded *and* monotonic, then it must be convergent.

bdd above. & increasing
bdd below & decreasing

12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

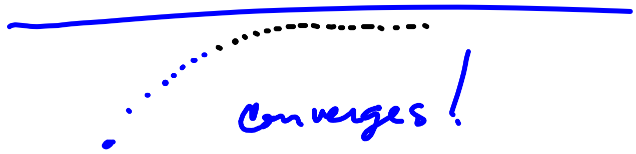
above & below.

The proof of Theorem 12 is based on the **Completeness Axiom** for the set of ℝ real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b.

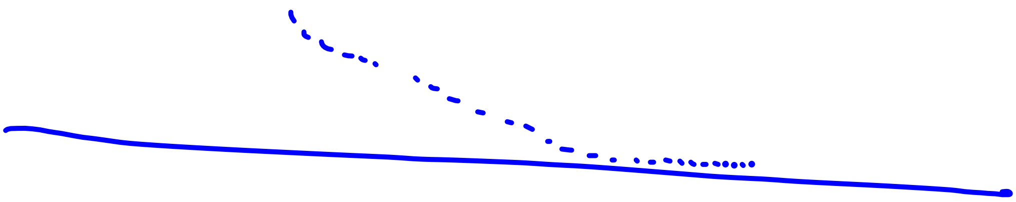
(This means that b is an upper bound for S, but if M is any other upper bound, then $b \leq M$.)

The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

Increasing & Bounded above $\Longrightarrow$ Convergency



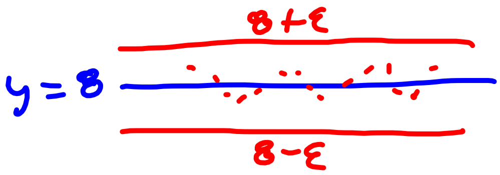
Bdd below & decreasing $\longrightarrow$ converges.

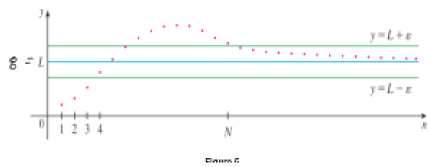


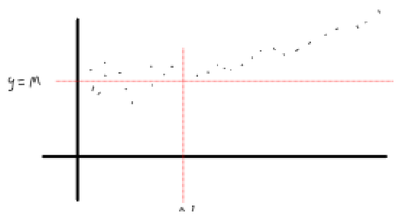
1. (a) What is a sequence?
   (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
   (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

a_n converges to 8.

Given $\varepsilon > 0$
$\exists N \ni n > N \Rightarrow |a_n - 8| < \varepsilon$.







2. (a) What is a convergent sequence? Give two examples.
(b) What is a divergent sequence? Give two examples.

(a) A convergent sequence is a sequence that comes arbitrarily close to a value, L, and STAYS close.

$$a_n = \frac{n+3}{n-7} \xrightarrow{n\to\infty} 1 = L$$

(b) A divergent sequence is a sequence that's not convergent!

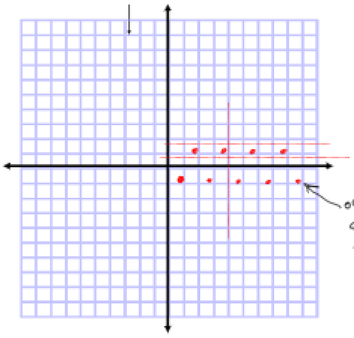
$$a_n = n \xrightarrow{n\to\infty} \infty \quad \text{(i.e., diverges!)}$$

$$a_n = (-1)^n n \xrightarrow{n\to\infty} \nexists$$

basically $\pm\infty$, depending on odds/evens.

$a_n = (-1)^n$

$\{-1, 1, -1, 1, \ldots\}$ diverges



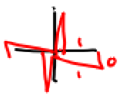
3–12 List the first five terms of the sequence.

3. $a_n = \dfrac{2n}{n^2 + 1}$

$\frac{2}{2} = 1$, $\frac{4}{5}$,

6. $a_n = \cos\dfrac{n\pi}{2}$

$\cos\frac{\pi}{2}$, $\cos\pi$



7. $a_n = \dfrac{1}{(n+1)!}$

0, −1, 0, 1, 0, −1, 0, 1, 0 . . .

10. $a_1 = 6, \quad a_{n+1} = \dfrac{a_n}{n}$

Recursive Definition

$6, \frac{6}{2}=3, \frac{3}{3}=1, \frac{1}{4}, \frac{\frac{1}{4}}{5} = \frac{1}{20}, \frac{\frac{1}{20}}{6} = \frac{1}{120}$

1 2 3 4 5

13–18 Find a formula for the general term a_n of the sequence assuming that the pattern of the first few terms continues.

13. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$

16. $\{5, 8, 11, 14, 17, \ldots\}$

$5+3(n-17)$

18. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

$\boxed{\sin\left(\frac{n\pi}{2}\right)\ ?}$ Yup

w/o #6, I'd've been thinking

6. $a_n = \cos\frac{n\pi}{2}$

$$\begin{cases} 0 & \text{if } n = 2K \quad \text{Evens} \\ (-1)^{K-1} & \text{if } n = 2K-1 \quad \text{odds} \end{cases}$$

$n = 1$
$(-1)^{K-1} = 1,$

$1 = 2(1) - 1 \Rightarrow K = 1$
$(-1)^{1-1} = -1^0 = 1$

$n = 3$
$(-1)^{3-1} = (-1)^2 = 1$

$n = 3 = 2(2) - 1 \Rightarrow K = 2$
$(-1)^{2-1} = (-1)^1 = -1$

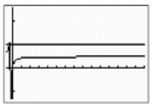
19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

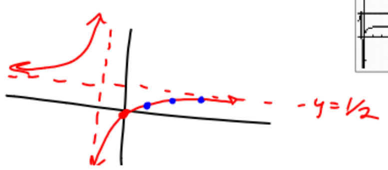
!?

19. $a_n = \dfrac{3n}{1 + 6n}$ **20.** $a_n = 2 + \dfrac{(-1)^n}{n}$

Hack wit' 'dis.

$$f(x) = \frac{3x}{6x+1} = \frac{3x}{6\left(x+\frac{1}{6}\right)} = \frac{x}{2\left(x+\frac{1}{6}\right)}$$





21. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

22. $a_n = 1 + \dfrac{10^n}{9^n}$

23–56 Determine whether the sequence converges or diverges. If it converges, find the limit.

23. $a_n = 1 - (0.2)^n \xrightarrow{n \to \infty} 1$

$(0.2)^n \to 0$

24. $a_n = \dfrac{n^3}{n^3 + 1} \xrightarrow{n \to \infty} 1$

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25. $a_n = \dfrac{\cancel{} + 5n^2}{\cancel{} + n^2} \xrightarrow{n \to \infty} 5$

26. $a_n = \dfrac{n^3}{n + 1} \xrightarrow{n \to \infty} \infty$

or just say "DIVERGES"

27. $a_n = e^{1/n}$

28. $a_n = \dfrac{3^{n+2}}{5^n}$