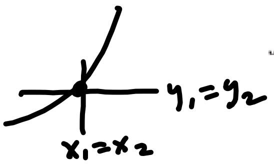
1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$



$$\boxed{f(x_1) = f(x) \Longrightarrow x_1 = x_2}$$

use this to show 1-to-1.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

$\boxed{2}$ **Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

3

$$f^{-1}(x) = y \iff f(y) = x$$

4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

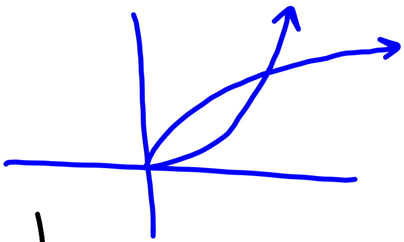
Step 2 Solve this equation for x in terms of y (if possible).

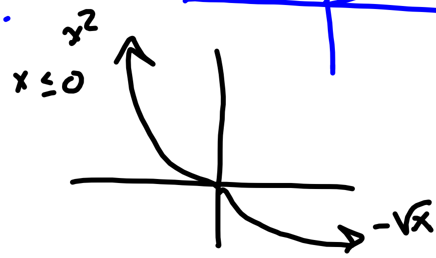
Step 3 To express f^{-1} as a function of x, interchange x and y The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

6 Theorem If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

6.5 If f is increasing, then f^{-1} is increasing.





7 Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Slick, when $f^{-1}(x)$ is hard to calculate.

Solving $y = f(x)$ for x or $x = f(y)$ for y can be IMPOSSIBLE!

Showing f is 1-to-1:

Suppose $f(x_1) = f(x_2)$ and show this forces $x_1 = x_2$

See yesterday's notes.

$f(x) = x^2 - 4x$, $x \geq 2$ is 1-to-1

$\mathcal{D}(f) = [2, \infty) = \mathcal{R}(f^{-1})$

$\mathcal{R}(f) = [-4, \infty) = \mathcal{D}(f^{-1})$

Find f^{-1}:

$y = x^2 - 4x$

$1y^2 - 4y + 2^2 = x + 4$

$\frac{4}{2} = 2 \rightsquigarrow 2^2$

$(y-2)^2 = x + 4$

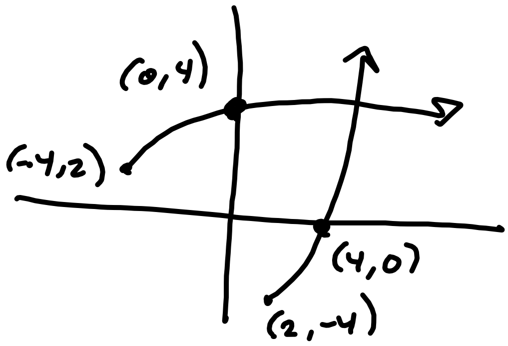
$\sqrt{(y-2)^2} = \sqrt{x+4}$

$|y-2| = \sqrt{x+4}$

$y - 2 = \pm\sqrt{x+4}$

$y = \pm\sqrt{x+4} + 2$

$y = \boxed{\sqrt{x+4} + 2 = f^{-1}(x)}$



$\mathcal{D}(f)$!

How did I know to take $+\sqrt{x+4}$ not $-\sqrt{x+4}$?

$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = [2, \infty)$ means

$y = \sqrt{x+4} + 2$ is f^{-1}.

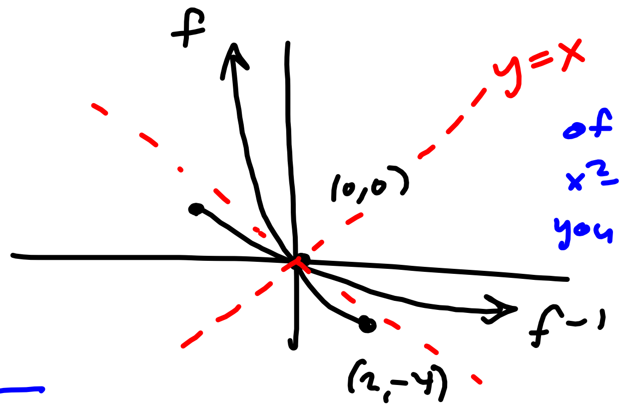
$f(x) = x^2-4x, \quad x \le 2 \Longrightarrow \mathcal{D}(f) = (-\infty, 2]$

$y^2-4y = x$

⋮

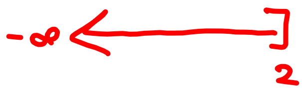
$(y-2)^2 = x+4$

⋮



This version of f^{-1} for x^2-4x will help you see $\left(\frac{1}{3}\right)^x$ & $\log_{\frac{1}{3}}(x)$ as inverses in the sequel.

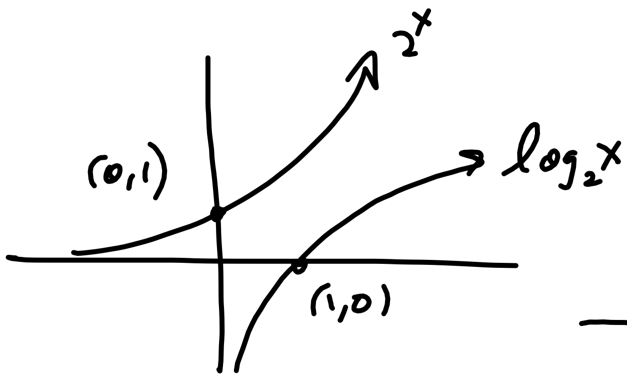
$y-2 = \pm\sqrt{x+4}$

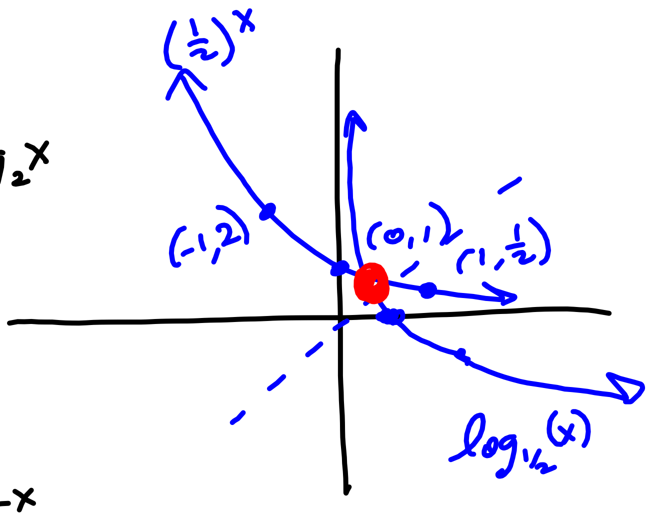


$y = \pm\sqrt{x+4} + 2$

want $\mathcal{R}(f^{-1}) = \mathcal{D}(f) = (-\infty, 2]$, so take

$y = -\sqrt{x+4} + 2 = f^{-1}(x)$





$$\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}$$

Showing f is <u>NOT</u> 1-to-1 is <u>subtle</u>.

Set $f(x_1) = f(x_2)$, but then show that $x_1 \neq x_2$ can still be true.

$x^2 - 4x$ is not 1-to-1

$x_1^2 - 4x_1 = x_2^2 - 4x_2$

$x_1^2 - 4x_1 + 2^2 = x_2^2 - 4x_2 + 2^2$

$\sqrt{(x_1-2)^2} = \sqrt{(x_2-2)^2}$

$x_1 - 2 = \pm (x_2 - 2)$ ← $|x_1 - 2| = |x_2 - 2|$

$x_1 - 2 = x_2 - 2$ OR $x_1 - 2 = -(x_2 - 2)$

$x_1 = x_2$ $x_1 - 2 = -x_2 + 2$ → Scrat

$x_1 = -x_2 + 4$

Not 1-to-1. Use this to NAIL the counterexample.

Let $x_1 = 1$.

$1 = 1$

#s 35-42

$1 = -x_2 + 4$

$-3 = -x_2$

$3 = x_2$

$1^2 - 4(1) = -3$

$3^2 - 4(3) = -3$

This says $(1,-3)$ & $(3,-3)$ are points on the graph of f.

8/26 Solutions for §6.1 are posted.

#s 39-42 are easier to do than they appear.
(Guess $f^{-1}(a) = 0$ or 1)

40 $f(x) = x^3 + 3\sin x + 2\cos x \quad a = 2$

$f(x) = a \Rightarrow x = 0$ is a solution.

#s 39-42 Find $(f^{-1})'(a)$

41 I think I messed up.

$\boxed{T7}$ If $f(x)$ is differentiable and 1-to-1, with $f(b)=a$. Then $b=f^{-1}(a)$ and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}$$

#41 $f(x) = x^2 + \tan\left(\frac{\pi}{2}x\right) + 3$

$-1 < x < 1$, $a=3$ → Input to $(f^{-1})'$

Want to know what x-value is sent to a. (Find b)

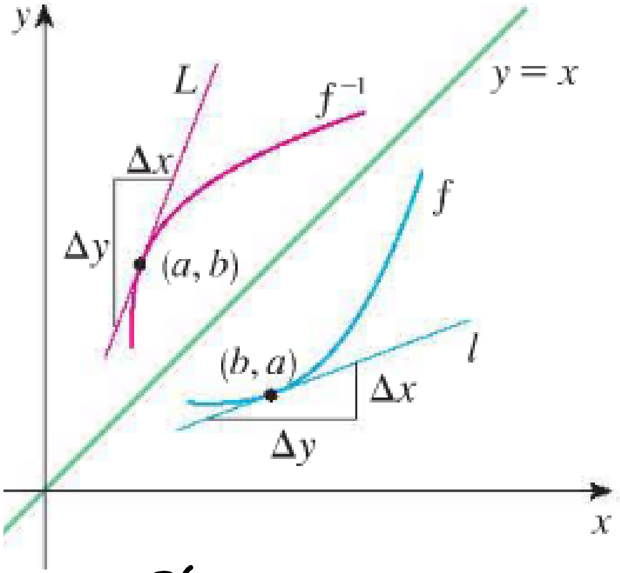
Set $\boxed{x^2 + \tan\left(\frac{\pi}{2}x\right) + 3 = 3}$ & solve ⇒ $x = 0 = b = f^{-1}(a)$

$(a,b) = (a, f^{-1}(a))$

$f'(x) = 2x + \frac{\pi}{2}\sec^2\left(\frac{\pi}{2}x\right)$

$f'(f^{-1}(a)) = f'(b) = 2(0) + \frac{\pi}{2}\sec^2\left(\frac{\pi}{2}(0)\right) = \frac{\pi}{2}$

$\longrightarrow (f^{-1})'(3) = \dfrac{1}{\frac{\pi}{2}} = \boxed{\dfrac{2}{\pi}}$



$$\frac{1}{f'(f^{-1}(a))} = (f^{-1})'(a)$$

(b,a) is point on graph of f

(a,b) is point on graph of f^{-1}

§6.2 #s 3, 5, 7-9*, 15-55 odds, 60, 65

THE MEAT starts on Pg 395

* $-2^{-x}=y$. How would you graph $y=(-2)^{-x}$?

You don't know squat about a negative base.

For Next time, Rough In (Annotate) §6.2.

#28

$$x^2+bx+c=0$$

$$x^2+bx=-c$$

$$x^2+bx+\left(\frac{b}{2}\right)^2=-c+\left(\frac{b}{2}\right)^2$$

$$\left(x+\frac{b}{2}\right)^2=\frac{b^2-4c}{4}$$

$$y=2x^2-8x \quad x\geq 2 \rightarrow \mathcal{D}(f)=\boxed{[2,\infty)=\mathcal{R}(f^{-1})}$$

$$y=2(x^2-4x)=y$$

$$x^2-4x=\frac{1}{2}y$$

$$x^2-4x+2^2=\frac{y}{2}+4$$

$$(x-2)^2=\frac{y+8}{2}$$

$$2y^2-8y=x$$

$$2(y^2-4y)=x$$

$$y^2-4y=\frac{1}{2}x$$

$$y^2-4y+2^2=\frac{1}{2}x+4$$

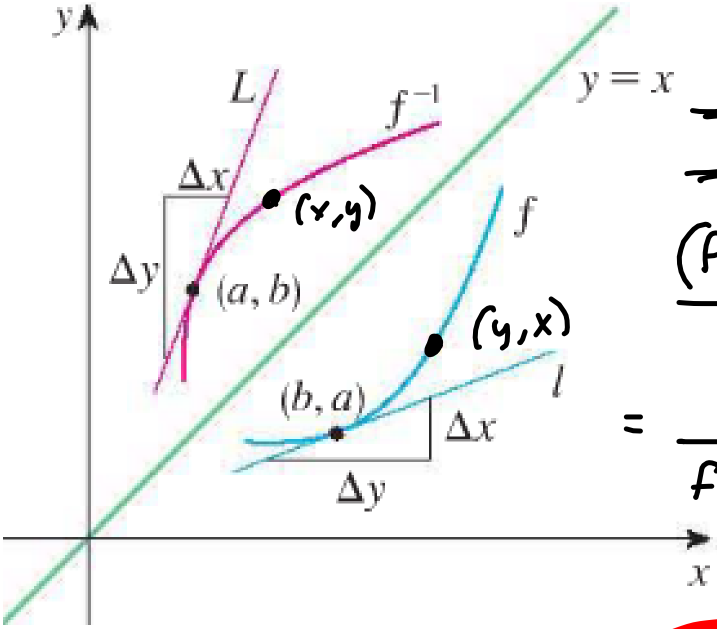
$$(y-2)^2=\frac{x+8}{2}$$

$$y-2=\pm\sqrt{\frac{x+8}{2}}$$

$$y=2\pm\sqrt{\frac{x+8}{2}}$$

$$y=2+\sqrt{\frac{x+8}{2}}$$ by $\mathcal{D}(f)=\mathcal{R}(f^{-1})$

0



$\boxed{T7}$ Proof

f is differentiable

$\Rightarrow$ f is continuous

$\Rightarrow f^{-1}$ .. ..

$$\frac{(f^{-1})(x) - (f^{-1})(a)}{x-a}$$

$$= \frac{y-b}{f(y)-f(b)}$$

By continuity $x \to a$ means $y \to b$.

$$= \frac{1}{\frac{f(y)-f(b)}{y-b}} \xrightarrow[\text{(i.e., } y \to b)]{x \to a} \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

$$\boxed{\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = f'(x)}$$

The definition used in the proof.