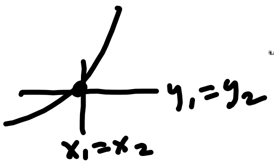
1 **Definition** A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$



$$\boxed{f(x_1) = f(x) \Rightarrow x_1 = x_2}$$

use this to show 1-to-1.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

$\boxed{2}$ **Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

3

$$\boxed{f^{-1}(x) = y \iff f(y) = x}$$

4

$$\boxed{\begin{array}{ll} f^{-1}(f(x)) = x & \text{for every } x \text{ in } A \\ f(f^{-1}(x)) = x & \text{for every } x \text{ in } B \end{array}}$$

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

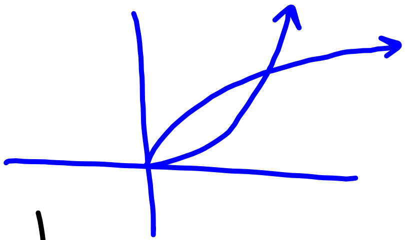
Step 2 Solve this equation for x in terms of y (if possible).

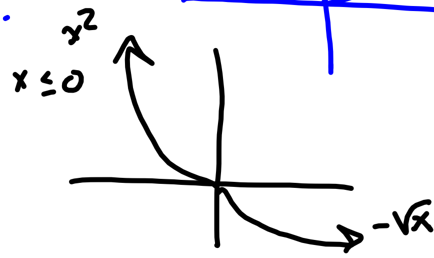
Step 3 To express f^{-1} as a function of x, interchange x and y The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

6 Theorem If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

6.5 If f is increasing, then f^{-1} is increasing.





7 Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Slick, when $f^{-1}(x)$ is hard to calculate.

Solving $y = f(x)$ for x or $x = f(y)$ for y can be IMPOSSIBLE!

Showing f is 1-to-1:

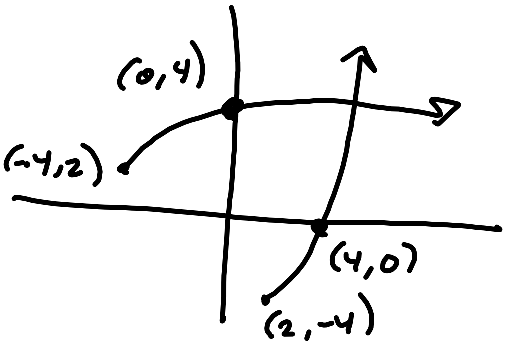
Suppose $f(x_1) = f(x_2)$ and show this forces $x_1 = x_2$

See yesterday's notes.

$f(x) = x^2 - 4x, \; x \geq 2$ is 1-to-1

$\mathcal{D}(f) = [2, \infty) = \mathcal{R}(f^{-1})$

$\mathcal{R}(f) = [-4, \infty) = \mathcal{D}(f^{-1})$



Find f^{-1}:

$y = x^2 - 4x$

$1y^2 - 4y + 2^2 = x + 4$

$\frac{4}{2} = 2 \rightsquigarrow 2^2$

$(y-2)^2 = x + 4$

$\sqrt{(y-2)^2} = \sqrt{x+4}$

$|y-2| = \sqrt{x+4}$

$y - 2 = \pm\sqrt{x+4}$

$y = \pm\sqrt{x+4} + 2$

$y = \boxed{\sqrt{x+4} + 2 = f^{-1}(x)}$

$\mathcal{D}(f)$!

How did I know to take $+\sqrt{x+4}$ not $-\sqrt{x+4}$?

$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = [2, \infty)$ means $y = \sqrt{x+4} + 2$ is f^{-1}.

$f(x) = x^2 - 4x, \quad x \le 2 \Longrightarrow \mathcal{D}(f) = (-\infty, 2]$

$y^2 - 4y = x$

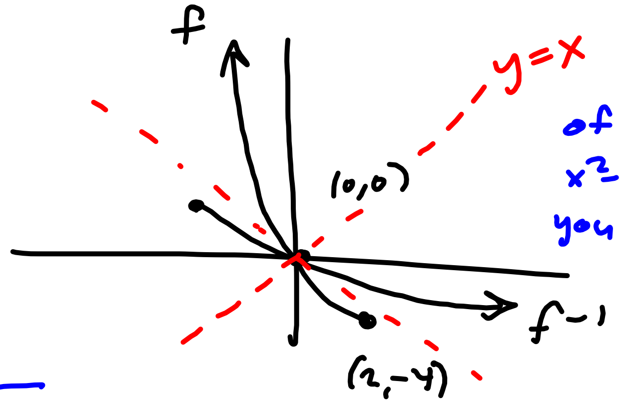
$\vdots$

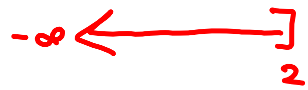
$(y-2)^2 = x + 4$

$\vdots$

$y - 2 = \pm\sqrt{x+4}$

$y = \pm\sqrt{x+4} + 2$

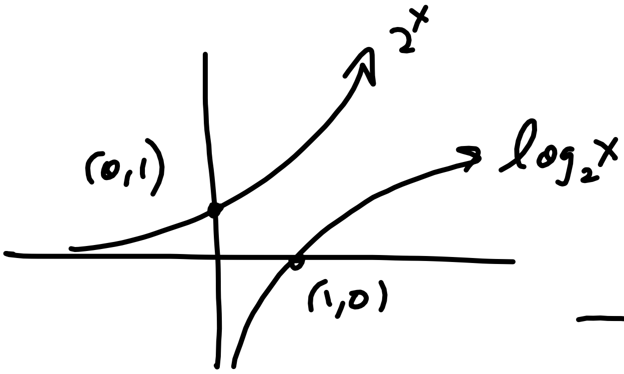


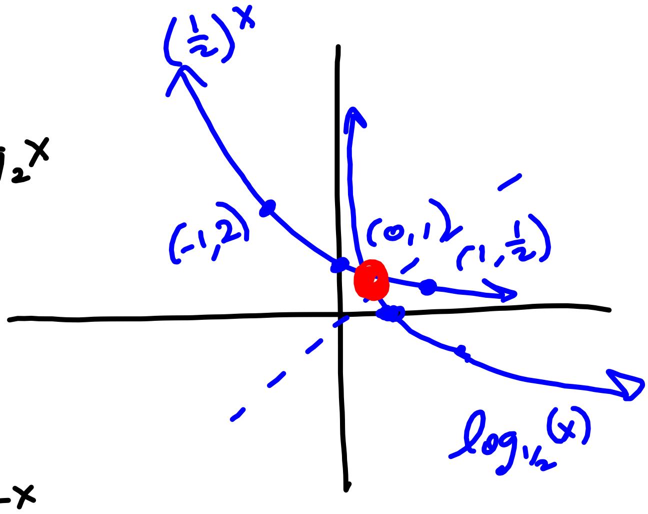


This version of f^{-1} for x^2-4x will help you see $\left(\frac{1}{3}\right)^x$ & $\log_{\frac{1}{3}}(x)$ as inverses in the sequel.

want $\mathcal{R}(f^{-1}) = \mathcal{D}(f) = (-\infty, 2]$, so take

$y = -\sqrt{x+4} + 2 = f^{-1}(x)$





$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

Showing f is NOT 1-to-1 is subtle.

Set $f(x_1)=f(x_2)$, but then show that $x_1 \neq x_2$ can still be true.

x^2-4x is not 1-to-1

$x_1^2-4x_1 = x_2^2-4x_2$

$x_1^2-4x_1+2^2 = x_2^2-4x_2+2^2$

$\sqrt{(x_1-2)^2} = \sqrt{(x_2-2)^2}$

$|x_1-2| = |x_2-2|$

$x_1-2 = \pm(x_2-2)$

$x_1-2 = x_2-2$ OR $x_1-2 = -(x_2-2)$

$x_1 = x_2$ $x_1-2 = -x_2+2$

$x_1 = -x_2+4$ Scrat

Not 1-to-1. Use this to NAIL the counterexample

#s 35-42

Let $x_1 = 1$.

$1=1$

$1 = -x_2+4$

$-3 = -x_2$

$3 = x_2$

$1^2-4(1) = -3$

$3^2-4(3) = -3$

This says $(1,-3)$ & $(3,-3)$ are points on the graph of f.