1. $y' + \cos x = y \quad \Rightarrow \quad y' + (-1)y = -\cos x$ is linear since it can be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

2. $y' + \cos y = \tan x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$. [$\cos y$ is not of the form $P(x)\,y$.]

3. $yy' + xy = x^2 \quad \Rightarrow \quad y' + x = x^2/y \quad \Rightarrow \quad y' - x^2/y = -x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

4. $xy + \sqrt{x} = e^x y' \quad \Rightarrow \quad y' = xye^{-x} + \sqrt{x}\,e^{-x} \quad \Rightarrow \quad y' + (-xe^{-x})y = \sqrt{x}\,e^{-x}$ is linear since it can be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2\,dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y' + 2e^{2x}y = 2e^{3x} \quad \Rightarrow \quad (e^{2x}y)' = 2e^{3x} \quad \Rightarrow \quad e^{2x}y = \int 2e^{3x}\,dx \quad \Rightarrow \quad e^{2x}y = \frac{2}{3}e^{3x} + C \quad \Rightarrow \quad y = \frac{2}{3}e^x + Ce^{-2x}$.

6. $y' = x + 5y \quad \Rightarrow \quad y' - 5y = x$. $I(x) = e^{\int P(x)\,dx} = e^{\int(-5)\,dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y' - 5e^{-5x}y = xe^{-5x} \quad \Rightarrow \quad (e^{-5x}y)' = xe^{-5x} \quad \Rightarrow \quad e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$ [by parts] $\quad \Rightarrow \quad y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$.

8. $x^2y' + 2xy = \cos^2 x \quad \Rightarrow \quad y' + \dfrac{2}{x}y = \dfrac{\cos^2 x}{x^2}$. $\quad I(x) = e^{\int P(x)\,dx} = e^{\int 2/x\,dx} = e^{2\,\ln|x|} = e^{\ln(x^2)} = x^2$.

Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y'. We rewrite it as $(x^2y)' = \cos^2 x$. Thus,

$$x^2y = \int \cos^2 x\,dx = \int \tfrac{1}{2}(1 + \cos 2x)\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \quad \Rightarrow$$

$$y = \frac{1}{2x} + \frac{1}{4x^2}\sin 2x + \frac{C}{x^2} \text{ or } y = \frac{1}{2x} + \frac{1}{2x^2}\sin x\cos x + \frac{C}{x^2}.$$

13. $(1+t)\,\dfrac{du}{dt} + u = 1 + t,\ t > 0$ [divide by $1+t$] $\quad \Rightarrow \quad \dfrac{du}{dt} + \dfrac{1}{1+t}u = 1$ ($\star$), which has the form $u' + P(t)\,u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)\,dt} = e^{\int [1/(1+t)]\,dt} = e^{\ln(1+t)} = 1 + t$.

Multiplying ($\star$) by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]' = 1 + t$. Thus,

$$(1+t)u = \int (1+t)\,dt = t + \tfrac{1}{2}t^2 + C \quad \Rightarrow \quad u = \frac{t + \frac{1}{2}t^2 + C}{1+t} \text{ or } u = \frac{t^2 + 2t + 2C}{2(t+1)}.$$

17. $\dfrac{dv}{dt} - 2tv = 3t^2e^{t^2},\ v(0) = 5$. $\quad I(t) = e^{\int(-2t)\,dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get

$$e^{-t^2}\frac{dv}{dt} - 2te^{-t^2}v = 3t^2 \quad \Rightarrow \quad \left(e^{-t^2}v\right)' = 3t^2 \quad \Rightarrow \quad e^{-t^2}v = \int 3t^2\,dt = t^3 + C \quad \Rightarrow \quad v = t^3e^{t^2} + Ce^{t^2}.$$

$5 = v(0) = 0 \cdot 1 + C \cdot 1 = C$, so $v = t^3e^{t^2} + 5e^{t^2}$.

20. $(x^2+1)\,\dfrac{dy}{dx}+3x(y-1)=0 \quad\Rightarrow\quad (x^2+1)\,y'+3xy=3x \quad\Rightarrow\quad y'+\dfrac{3x}{x^2+1}\,y=\dfrac{3x}{x^2+1}$.

$I(x)=e^{\int 3x/(x^2+1)\,dx}=e^{(3/2)\ln|x^2+1|}=\left(e^{\ln(x^2+1)}\right)^{3/2}=(x^2+1)^{3/2}$. Multiplying by $(x^2+1)^{3/2}$ gives

$(x^2+1)^{3/2}\,y'+3x(x^2+1)^{1/2}\,y=3x(x^2+1)^{1/2} \quad\Rightarrow\quad \left[(x^2+1)^{3/2}y\right]'=3x(x^2+1)^{1/2} \quad\Rightarrow$

$(x^2+1)^{3/2}\,y=\int 3x(x^2+1)^{1/2}\,dx=(x^2+1)^{3/2}+C \quad\Rightarrow\quad y=1+C(x^2+1)^{-3/2}$. Since $y(0)=2$, we have

$2=1+C(1) \quad\Rightarrow\quad C=1$ and hence, $y=1+(x^2+1)^{-3/2}$.

23. Setting $u=y^{1-n}$, $\dfrac{du}{dx}=(1-n)\,y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx}=\dfrac{y^n}{1-n}\dfrac{du}{dx}=\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation

becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}+P(x)u^{1/(1-n)}=Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx}+(1-n)P(x)u=Q(x)(1-n)$.

24. Here $xy'+y=-xy^2 \quad\Rightarrow\quad y'+\dfrac{y}{x}=-y^2$, so $n=2$, $P(x)=\dfrac{1}{x}$ and $Q(x)=-1$. Setting $u=y^{-1}$, u satisfies

$u'-\dfrac{1}{x}u=1$. Then $I(x)=e^{\int(-1/x)dx}=\dfrac{1}{x}$ (for $x>0$) and $u=x\left(\displaystyle\int\frac{1}{x}\,dx+C\right)=x(\ln|x|+C)$. Thus,

$y=\dfrac{1}{x(C+\ln|x|)}$.

26. $xy''+2y'=12x^2$ and $u=y' \quad\Rightarrow\quad xu'+2u=12x^2 \quad\Rightarrow\quad u'+\dfrac{2}{x}\,u=12x$.

$I(x)=e^{\int(2/x)\,dx}=e^{2\ln|x|}=\left(e^{\ln|x|}\right)^2=|x|^2=x^2$. Multiplying the last differential equation by x^2 gives

$x^2u'+2xu=12x^3 \quad\Rightarrow\quad (x^2u)'=12x^3 \quad\Rightarrow\quad x^2u=\int 12x^3\,dx=3x^4+C \quad\Rightarrow\quad u=3x^2+C/x^2 \quad\Rightarrow$

$y'=3x^2+C/x^2 \quad\Rightarrow\quad y=x^3-C/x+D$.